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Information Flows and Memory in Games

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Abstract

We propose that the mathematical representation of situations of strategic interactions, i.e., of games, should separate the description of the rules of the game from the description of players' personal traits. Yet, we note that the standard extensive-form partitional representation of information in sequential games does not comply with this separation principle. We offer an alternative representation that extends to all (finite) sequential games the approach adopted in the theory of repeated games with imperfect monitoring, that is, we describe the *flow* of information accruing to players rather than the *stock* of information retained by players, as encoded in information partitions. Mnemonic abilities can be represented independently of games. Assuming that players have perfect memory, our flow representation gives rise to information partitions satisfying perfect recall. Different combinations of rules about information flows and of players mnemonic abilities may give rise to the same information partitions. All extensive-form representations with information partitions, including those featuring absentmindedness, can be generated by some such combinations.

1 Introduction

All games whose play can be impemented on IT platforms—including market games, auctions, games played in the lab, and games like poker and chess—are defined by formal rules clearly specifying (i) the feasible alternatives of active players according to previous play, (ii) what information accrues to players, and (iii) the material consequences of each complete play (terminal sequence of actions and, possibly, realizations of chance moves). Such rules should be amenable to a description using a formal mathematical language and that this description should be independent of the personal features of the agents playing the game in each role in any particular instance of play. Indeed, we propose as a general methodological tenet the following **separation principle**: *the formal description of the rules of the game should be independent of the personal features of those who happen to play the game.*

This is easily done as far as the concerned personal features are related to taste, or preferences. For example, in simultaneous-move games with monetary consequences, one can describe preferences over monetary lotteries as von Neumann-Morgenstern utility functions, under a set of well understood behavioral assumptions. Appending such utility functions to the formal description of the rules of the game and adding assumptions about players' interactive knowledge (or interactive beliefs) about the relevant aspects of the situation of strategic interaction, one obtains a mathematical structure amenable to game-theoretic analysis.[1]

Yet, as we consider games with sequential moves, it is not obvious that the extant game-theoretic formalism complies with the aforementioned separation principle. Indeed, with very few notable exceptions concerning multistage games (such as Myerson, 1986, and Myerson & Reny, 2020, whose purposes are different from ours), the information that accrues to players as the play unfolds is described by information partitions of the set of partial plays (or nodes): two partial plays x and y are indistinguishable by player i if and only if they belong to the same cell (equivalence class) of her information partition.[2] Furthermore, it is usually assumed that information partitions satisfy a property called "perfect recall". The interpretation of this property is that a player remembers all the information previously provided to her and all the actions previously taken by her.[3] This clarifies that a player's information partition represents a kind of *stock* of information at every relevant node.[4] Put differently, an information partition provides

[1]See, e.g., Osborne and Rubinstein (1994), or Battigalli, Catonini, and De Vito (2021).

[2]Of course, we are considering work on general games. The literature on repeated games with imperfect monitoring (e.g., Mailath & Samuelson, 2006) provides a whole class of "exceptions", that is, works that describe information in compliance with the separation principle.

[3]In the words of Kuhn (1953, p. 213), perfect recall is "equivalent to the assertion that each player is allowed by the rules of the game to remember everything he knew at previous moves and all of his choices at those moves". However, it is not clear at all how the rules of the game could *prevent* players from remembering things they did and observed, as memory is a *subjective* attribute of players – this seems to suggest a potential violation of our separation principle.

[4]We use "relevant" loosely to refer to all the nodes for which a given player's information matters, which include all the nodes where she is active.

"snapshots" of a player's information at each point of the game. Yet, the formalism is crucially silent on where such information comes from: should this be interpreted as the information that this player is able to retain and use, or as an objective representation of the cumulated information that accrued to her? The former interpretation implies that information partitions are hybrid representations mixing objective features of the rules of the game with a player's personal cognitive abilities, thus violating the separation principle. The latter interpretation is possible if information partitions have the perfect recall property, provided one maintains that the agents who happen to play the game have perfect mnemonic abilities.

In this paper we put forward and analyze a general mathematical description of the rules of (finite) games, whereby we represent the *flows* of information accruing to players. Specifically, we assume that throughout the game players observe some signals informing them of the play unfolding. For instance, in a game played in a lab or on an online platform a mediator may provide players with some details about how the game has been played and with some instructions about how to proceed (the feasible actions). The sequence of actions taken and signals observed form the stream of information potentially available to players. Importantly, this description is independent of players' personal features, thus complying with the separation principle.

We obtain formal relationships between our "flow" representation and the traditional "stock" representation with information partitions, and we show in passing that a game with information partitions (satisfying perfect recall) is consistent with different flow representations, hence with different rules about the accrual of information to players. Under the informal assumption that players have perfect memory (and that this is common knowledge), such differences are immaterial. We also prove that an information partition satisfies perfect recall if and only if it can be obtained from a flow-based description of a game, under the as yet informal assumption that players have perfect memory (Proposition 4). We first develop our analysis for the case of multistage games, which is simpler and easier to grasp. Next we generalize to all finite games. This sequence implies some redundancies, but we find it pedagogically useful.

We also put forward a *game-independent*, rudimentary analysis of *memory*. In compliance with the separation principle, this allows us to combine a flow-based description of a game with a formal specification of players' personal mnemonic abilities. An analogy may help clarify our position: just like risk attitudes are framed within the theory of choice and only subsequently embedded in game-theoretic analysis, we believe that a formalization of agents' mnemonic abilities should come from a suitable "theory of memory" conceptually unrelated (although complementary) to game theory. We follow this route by appending to a given game described with information flows a specification of players' mnemonic features and thus obtain a game with possibility correspondences[5] (which may or may not be information partitions), just like appending to a game form a specification of players' preferences over lotteries allows the derivation of the utilities of terminal nodes.[6]

This explicit account of memory allows us to make the aforementioned informal assumption of "perfect memory" precise (Proposition 5): when the cognitive features of the agents who happen to play the game satisfy "perfect memory", we obtain information partitions satisfying perfect recall from our flow-based description of the game. Furthermore, we prove that *any* standard information partition (even those that fail perfect recall) can be retrieved in our setting by considering a suitable combination of information flows and descriptions of memory (Proposition 6): this makes our approach at least as expressive as the traditional one.

Our flow approach also naturally lends itself to the analysis of issues concerning the information of inactive players, which is key – for example – in the theories of self-confirming equilibrium and of psychological games.[7] Finally, we elucidate "absentmindedness", i.e., the possibility that a player forgets not just what actions she took, but also whether she took some action at all (see, e.g., Chapter 11 in Osborne & Rubinstein, 1994). According to our approach, absentmindedness is the consequence of quite natural personal cognitive limitations that our analysis of memory can easily capture.

Roadmap The remainder of this paper is organized as follows. Section 2 introduces some notation. Section 3 proposes our flow-based description of the rules of interactions for multistage games. Section 4 illustrates the conventional approach to modeling multistage games. Section 5 presents results relating the two approaches. Section 6 generalizes the framework allowing for non-multistage structures. Section 7 explores the possibility of explicitly describing players' ability to recall the pieces of information they

[5]This means that the formal description of a player's memory allows us to identify the partial or complete plays (nodes) that are deemed *possible* by such player at some point of the game, based on what she remembers.

[6]Misleadingly called "payoffs" in the technical jargon of game theory.

[7]On psychological games, see the survey by Battigalli and Dufwenberg (forthcoming). The self-confirming equilibrium idea was independently put forward by several authors, including Fudenberg and Levine (1993), who coined the term. See the literature review in the Discussion section of Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2015).

are exposed to. Section 8 concludes and discusses the related literature. Appendix A collects proofs.

2 Preliminaries

Maintained assumptions In the following, we assume that the set of players I and the sets of actions A_i ($i \in I$) potentially available to each player i are *finite*. Furthermore, we also assume that the game always terminates after a finite sequence of actions, or profiles of simultaneously chosen actions.[8]

Sequences For a generic set X and $n \in \mathbb{N}$, we let $X^n := \bigtimes_{k=1}^{n} X$ denote the n-fold Cartesian product of X, with generic element $x^n = (x_k)_{k=1}^n$. By convention, we let $X^0 := \{\varnothing_X\}$ denote the singleton with the **empty sequence** of elements of X as its unique element; we often drop the subscript and just write $\varnothing$ when this causes no confusion. The empty set is denoted by the (different) symbol $\emptyset$. For $N \in \mathbb{N}$, $X^{\leq N} := \bigcup_{n=0}^{N} X^n$ denotes the set of sequences of elements of X of length N or less. While superscripts are used to denote the length of a given sequence (i.e., we write x^n to denote a generic element of X^n), we often use Greek letters to denote sequences whose length is left unspecified. Hence, ξ is a generic element of $X^{\leq N}$. For each $\xi \in X^{\leq N}$, we let $\ell(\xi)$ denote the **length** of ξ. Finally, we let $\preceq$ denote the reflexive "prefix of" relation defined as follows: for each $\alpha = (a_k)_{k=1}^m$ and $\beta = (b_k)_{k=1}^n$ in $X^{\leq N}$, $\alpha \preceq \beta$ if $m \leq n$ and $a_k = b_k$ for each $k \in \{1, \dots, m\}$.

Trees Fix a generic set X and consider $V \subseteq X^{\leq N}$ (with $N \in \mathbb{N}$). We say that V is a **tree** if, whenever $\beta \in V$ and $\alpha \preceq \beta$, then $\alpha \in V$ – i.e., V is a tree if it is closed under the "prefix of" relation $\preceq$ (cf. Kechris, 1995, Definition 2.1).[9] Given a tree $V \subseteq X^{\leq N}$, for each element $\xi \in V$, we are going to denote as $\operatorname{pre}\xi$ the **immediate predecessor**, or **parent**, of ξ, that is, the unique element of V such that $\ell(\operatorname{pre}\xi) = \ell(\xi) - 1$ and $\operatorname{pre}\xi \preceq \xi$.

Notation We denote correspondences with calligraphic letters and collections of sets with script letters. We try to make our notation suggestive,

[8] The finiteness assumption is only a simplification. Our approach extends seamlessly to infinite games (cf. Myerson & Reny, 2020 for the special case of multistage games).

[9] Any tree V defined in this way can be naturally mapped into an equivalent tree defined in a graph-theoretic fashion. In particular, the set of vertices of such graph is isomorphic to V, and any two distinct vertices u, v are connected by a path if and only if $u \preceq v$ or $v \preceq u$.

and the symbols used to indicate the mathematical objects of interest will often be mnemonics for the role of such objects. Similarly, equations and logical statements in display mode will be associated with mnemonic labels.

3 Multistage game structures and flows of information

We call "game structure" the mathematical description of the rules of interaction, without the function mapping complete paths to outcomes (see Battigalli, Leonetti, & Maccheroni, 2020). We relate descriptions featuring "flows" of information to descriptions featuring "stocks" of information, represented by information sets. Here we start with the former, assuming a multistage structure.

We assume that, after each stage, players receive some messages about the previous play, in addition to observing their own actions. Such messages play a double role: on one hand, they (perhaps imperfectly) inform players of how the game has unfolded; on the other hand, they inform players of the actions they can take. For instance, in an ascending auction played on an online platform, a player may be notified right after the beginning that "the first bid was \$100: bid at least \$101 to continue". Or, more generally, messages could look like: "your opponent moved, now you can choose a or b", or "your opponent chose c, now you can choose a or b". The bottom line is that such messages can be more or less informative about the behavior of others, but they provide players with all the instructions needed to play the game (which requires, of course, knowing one's own feasible actions).

More specifically, we are going to assume in this first part that *all* players *simultaneously* receive some message before and after each stage is played. This is in line with the multistage restriction, as multistage games are precisely those in which players always "know" how many stages have been played (cf. Kuhn, 1953): in our setting, the messages accrual allows players to keep track of the stage the game is at. In the following, we illustrate the key ingredients of the analysis.

Players, actions, and messages We start by positing a finite set of **players** I, a maximal duration $T \in \mathbb{N}$ of the game, and, for each player $i \in I$, a finite set A_i of **actions** that will potentially be available to player i. As already mentioned, we assume that players receive some game-specific messages as the play unfolds: this is why we posit, for each player $i \in I$ and stage t, a finite set $M_{i,t}$ of **messages** that i can receive in stage t, with

$M_{i,s} \cap M_{i,t} = \emptyset$ if $s \neq t$, meaning that the message always reveals the stage just completed. With this, we let $A := \times_{i\in I} A_i$, $M_i = \bigcup_{k=0}^{T} M_{i,k}$ (where $M_{i,0}$ contains only the "start" message $m_{i,0}$), and $M := \times_{i\in I} M_i$ respectively denote the sets of action profiles, messages for i, and message profiles.

A rule to determine feasible actions We said that messages inform players of their feasible actions. In particular, players *need not remember* the actions they previously took or the messages they previously received to be able to understand what they can do: the message just received also encodes the set of feasible actions.

To formalize this idea, we introduce, for each player $i \in I$, an **action feasibility correspondence** $\mathcal{A}_i : M_i \rightrightarrows A_i$. Thus, $\mathcal{A}_i(m_i) \subseteq A_i$ ($i \in I$, $m_i \in M_i$) is the set of actions that player i can take after receiving message m_i. As a matter of terminology, we say that player $i \in I$ is **active** after message $m_i \in M_i$ if $|\mathcal{A}_i(m_i)| > 1$ – that is, if she can choose from at least two actions after observing message m_i. If a player has only one feasible action at some stage, it means that she is inactive and her only feasible action is to "wait". This dummy action is neglected in our notation. An empty set of feasible actions means "game over."

It is convenient to introduce the joint feasibility correspondence $\mathcal{A} : M \rightrightarrows A$, defined as $(m_i)_{i\in I} \mapsto \times_{i\in I} \mathcal{A}_i(m_i)$, that specifies which profiles of actions are feasible after some profile of messages is observed. We also assume that $\mathcal{A}((m_i)_{i\in I}) = \emptyset$ if and only if $\mathcal{A}_j(m_j) = \emptyset$ for all $j \in I$. That is, as soon as the game is over for some player (i.e., such player does not have any feasible action to take) it is over for everyone.

Note that this formulation assumes that players are always "alert", that is, they are always able to receive and process information. This assumption will be relaxed to generalize our framework to non-multistage games (cf. Section 6).

A rule to generate messages At this point, we need to discuss how messages are generated. Obviously such generative process make messages depend on the action profiles chosen in the game. With this in mind, we posit a profile of **individual feedback functions** $f_i := (f_i^t : A^t \to M_{i,t})_{t=0}^{T}$ ($i \in I$). Intuitively, for each $i \in I$ and $t \in \{1, \ldots, T\}$, $f_i^t : A^t \to M_{i,t}$ is player i's end-of-stage-t feedback function, which specifies which message player i would observe after any conceivable sequence of action profiles of length t. Recalling that $A^0 = \{\varnothing_A\}$, function $f_i^0 : A^0 \to M_{i,0}$ instead specifies the first message that player i receives at the beginning of the game: such message informs i that the game is starting, and it specifies (via the feasibility correspondence $\mathcal{A}_i$) i's initial set of feasible actions.

As for the feasibility correspondences, it is useful to derive a finite sequence of **collective feedback functions** $f := (f^t : A^t \to M_t)_{t=0}^T$, where, for each $t \in \{0, \ldots, T\}$, f^t is the map $a^t \mapsto (f_i^t(a^t))_{i \in I}$. In this first part on multistage games, the first message profile received by players at the root of the game is denoted as $m_0 := f^0(\varnothing_A)$. Note that each player's stage feedback may well depend on the action she just played, or on past actions of everyone. To make sense of our analysis, it is necessary to informally assume that either messages are explicit sentences in a language understood by players, or that each player knows at least her own feasibility correspondence and feedback function.

To give a better sense of the feedback functions, we introduce some terminology and we discuss some examples. We say that feedback is **perfect** if at each stage and for each player the feedback received allows that player to exactly infer the actions just chosen by others, regardless of the action she played. Formally, for each $i \in I$, $t \in \{1, \ldots, T\}$, $a^{t-1} \in A^{\leq T}$, and $a_i \in A_i$, $f_i^t(a^{t-1}, (a_i, \cdot))$ is injective.[10] We say that feedback is **cumulative** if new messages remind players of previously available pieces of information (i.e., actions chosen and messages received). Formally, for each pair of sequences of action profiles $a^t, b^t \in A^{\leq T}$ of the same length and for each player $i \in I$, $\mathrm{proj}_{A_i^{\leq T}}\, a^t \neq \mathrm{proj}_{A_i^{\leq T}}\, b^t$ or $f_i^t(a^t) \neq f_i^t(b^t)$ imply that, for each pair of successors $c^u \succ a^t$ and $d^u \succ a^t$, $f_i^u(c^u) \neq f_i^u(d^u)$. The condition says that, for each player, whenever two sequences of action profiles differ in the information they convey to that player (via either the actions chosen or the feedback observed), then also subsequent action profile sequences will result in different messages. To put it differently, future feedback incorporates past information, which includes information about own actions in previous stages. A couple of informal examples may help shed light on the terminology we use, as well as on the nature of feedback.

Example 1 (Chess) In chess, players observe all the moves performed at each stage, as well as the resulting positions of pieces on the chessboard. However, information about the previous play differs based on how and where the game is played.

Consider first a *friendly, in-person match* played by amateurs, where we assume that players do not write down the moves they take. In such setting, in each stage they observe the move of the active player and the end-of-stage

[10] Obviously, this does not involve function f_i^0 ($i \in I$), which determines the initial message, generated before any actions are chosen.

positions of pieces on the board, but they are not reminded of past play. If players recall some past moves, then it is just because they memorized them. Hence, feedback is perfect but not cumulative.

If instead the game is played in a *competitive tournament*, or on an *online platform* such as chess.com, the rules of the game themselves provide players with a complete account of the game unfolding. Indeed, in both such settings the log of moves taken throughout the game is publicly available. This makes feedback both perfect and cumulative.

Lastly, *blind chess* provides yet another feedback structure: in such game, the only available feedback pertains to the last move taken, and players have to remember past moves and figure out the positions of pieces on the board.[11] As in the first case, feedback is perfect but not cumulative. ▲

Example 2 (Auctions) Suppose that a number of agents repeatedly engage in first-price sealed-bid auctions. This may be due, for example, to the fact that multiple items are being sold. After each round of bids, players may have access to different forms of feedback. For instance, players may be informed only of the current-round winning bid: in such case feedback is neither perfect nor cumulative. If instead players are told the sequence of winning bids of all previous rounds, feedback is cumulative but not perfect. Specifically, "cumulativeness" comes from the fact that the feedback of a given stage incorporates previous feedback, as the communication of the previous-rounds winning bids is repeated as the interaction progresses. For feedback to be both perfect and cumulative, players must be informed after each round of *all* the bids made in previous rounds. ▲

Game structure We can now put things together, and we obtain the following definition.

Definition 1 *A* ***flow-based multistage game structure*** *is a tuple*

$$\Gamma_{FM} = \langle I, (A_i, M_i, f_i, \mathcal{A}_i)_{i \in I} \rangle.$$

Importantly, the foregoing discussion clarifies that all the mathematical objects forming a flow-based multistage game structure represent properties of the game that do not hinge in any way on the personal features of those

[11] Of course, the rules have to account for the possibility that a player with imperfect memory, or an imperfect ability to figure out the positions of pieces on the board, might attempt an illegal move. For example, they could stipulate that moves consist of instructions, every instruction can be given, and instructions to execute illegal moves terminate the game with the loss of the moving player.

who happen to play the game. To see this, note that it would be possible to give an explicit and objective description or definition of each of the objects in Definition 1 if one were to design, for example, a lab experiment.

Note that we can retrieve a tree H of histories of actions and messages in a constructive way:

$$\begin{aligned} H := \big\{ & (m_0, (a_k, m_k)_{k=1}^t) \in M \times (A \times M)^{\leq T} : \forall\, k \in \{1, \ldots, t\}, \\ & a_k \in \mathcal{A}(m_{k-1}),\ m_k = f^k(a^k) \big\} \cup \{\varnothing\}, \end{aligned}$$

where $\varnothing$ represents the empty history (the root of the game), and m_0 is the profile of initial messages received by players. One can check that H is indeed a tree. The elements of H are called **extended histories**.[12]

In the present framework, extended histories are sequences of profiles of *actions and messages*, where each player is assumed to observe only the action she chooses and the message she receives at the end of each stage. Importantly, this approach assumes that players are always alert (i.e., they are always able to receive and process information), as messages can be observed even when players are not active.

A generic extended history has the form $h = (m_0, (a_1, m_1), \ldots, (a_t, m_t))$, where the subscript denotes the stage at which an action profile is played or a message profile is received. Hence, each extended history features an initial profile of messages followed by a sequence of pairs of action and message profiles, one for each stage through which the extended history unfolds. This formalism highlights the double role of messages. On one hand, they inform players of their feasible actions, and hence an initial message profile m_0 is needed for players to realize what they can do at the beginning. On the other hand, messages give players information about previous play. In particular, a final message profile m_t lets players know the game is over and may give them some further information about the just completed play.

It is possible to derive from H the feasible sequences of action profiles that can be played according to the rules of the game. The set of such sequences is $P := \mathrm{proj}_{A^{\leq T}} H$, and its elements are called **plays**.[13] Of course,

[12]Our representation in terms of sequences of (messages and) actions is similar to that of Osborne and Rubinstein (1994), who use the term "history" to denote feasible sequences of (profiles of) actions. Here the adjective "extended" refers to the fact that our formalism includes profiles of messages in addition to profiles of actions.

[13]The term "play" has been used in the literature to refer to complete plays, which correspond to terminal nodes (cf. Kuhn, 1953). However, for the sake of clarity, we find it convenient to use the same term to include also partial plays, which correspond to non-terminal nodes.

the rules of the game allow play $(a_k)_{k=1}^t$, only if they allow every prefix $(a_k)_{k=1}^{\ell}$ ($\ell \leq k$). Therefore:

Remark 1 The set of possible plays $P \subseteq A^{\leq T}$ is a tree. ♦

As the game unfolds, players are informed only of the actions they take and of the messages they receive. To formalize this idea, for each $i \in I$, let $C_i := \mathrm{proj}_{M\times(A_i\times M_i)^{\leq T}} : M \times (A \times M)^{\leq T} \to M_i \times (A_i \times M_i)^{\leq T}$ be the projection map $(m_0, (a_j^t, m_j^t)_{j\in I}) \mapsto (m_{0,i}, (a_i^t, m_i^t))$. In words, $C_i(h)$ ($i \in I$, $h \in H$) is the personal history experienced by i given extended history h, and it can be interpreted as the *cumulated information* (i.e., actions played and messages received) player i has access to given h. We may also refer to $C_i(h)$ ($h \in H$) as the *stream of information, or personal history* experienced by player i within extended history h. The set of **personal histories** player i can experience as the game unfolds is defined as $H_i := C_i(H)$.

4 Stock description of information

In this section, we describe the conventional approach used to model information. In particular, Section 4.1 draws on Battigalli, Leonetti, and Maccheroni (2020) to give a standard definition of multistage game structure similar to that of Osborne and Rubinstein (1994) . Section 4.2 then explains how such formalism can be extended to facilitate the comparison with flow-based multistage game structures.

4.1 Standard information structures

We start with the definition of a game structure *à la* Osborne and Rubinstein (1994), whereby information is described by means of information sets, as is standard for the description of abstract sequential games since the seminal work of von Neumann and Morgenstern (1944). For a fixed tree $V \subseteq A^{\leq T}$ and all vertices/sequences $v \in V$ and players $i \in I$, let $A(v) := \{a \in A : (v, a) \in V\}$ and $A_i(v) := \{a_i \in A_i : \exists\, a_{-i} \in A_{-i}, (v, (a_i, a_{-i})) \in V\}$ (i.e., the projection of $A(v)$ onto A_i) respectively denote the sets of action profiles and actions of i consistent with V given v.

Definition 2 (cf. Osborne & Rubinstein, 1994, Definition 200.1) *A (finite)* ***standard multistage game structure*** *is a tuple*

$$\Gamma_{MS} = \langle I, V, (A_i, \mathscr{Q}_i)_{i\in I}\rangle,$$

where:

- $V \subseteq A^{\leq T}$ *is a* tree *(with vertices called "**plays**") satisfying:*

$$\forall\, v \in V, \quad A(v) = \bigtimes_{i \in I} A_i(v). \tag{APF}$$

- $\mathscr{Q}_i$ *is the collection of* ***information sets*** *of player i ($i \in I$), with $\mathscr{Q}_i$ being a partition of $\{v \in V : |\mathcal{A}_i(v)| > 1\} \cup \{\varnothing\} =: V_i \cup \{\varnothing\}$ that satisfies the following properties:*[14]

$$\forall\, v, w \in V_i, \quad \big(Q_i(v) = Q_i(w) \Longrightarrow A_i(v) = A_i(w)\big); \tag{KfA}$$
$$\forall\, v, w \in V_i, \quad \big(Q_i(v) = Q_i(w) \Longrightarrow \ell(v) = \ell(w)\big). \tag{KS}$$

In the definition above, $Q_i(v)$ ($i \in I$, $v \in V$) denotes the cell of $\mathscr{Q}_i$ that contains v. The same notation will be used subsequent sections.

A few comments are in order. First, the tree V specifies what sequences of action profiles can be played throughout the game, and, for each $v \in V$ and $i \in I$, $A_i(v) \subseteq A_i$ is the set of player i's available actions at v.[15] By definition, for each $v \in V$ and $a \in A$, we have that $(v, a) \in V$ if and only if $a \in A(v)$, and condition (APF) (action profile feasibility) imposes that what is feasible for i given v is logically independent of what is feasible for j given v. A play (or node, or vertex) v is terminal if and only if $A(v) = \emptyset$. Under such representation, being inactive at some play v amounts to having only one available action. The foregoing observations are consistent with the analysis of Section 3.

Second, an information set $Q_i \in \mathscr{Q}_i$ of player i is a set of plays where player i is active that player i cannot distinguish, and we refer to collection $\mathscr{Q}_i$ as the **(standard) information structure** of player i. Note that condition (KS) implies that $Q_i(\varnothing) = \{\varnothing\}$. That is, the information set containing the empty play is a singleton, and hence "being at the root of the game" is a situation that can be correctly recognized by players.

Third, condition (KfA) (knowledge of feasible actions) requires that two plays be indistinguishable for a player only if the set of actions available to such player after the two plays is the same – otherwise, players would not even able to play the game, as they might be unsure of the actions they can take at some point of the play. Condition (KS) (know-the-stage) further

[14]As a shorthand, for each $v \in V$ and $i \in I$, we let $Q_i(v)$ denote the unique $Q_i \in \mathscr{Q}_i$ such that $v \in Q_i$.

[15]Note, in this *axiomatic* approach, a (game) tree V is posited as primitive element of the analysis, and the feasible action profiles are derived from V. This should be contrasted with the more *constructive* approach of Definition 1, whereby action feasibility correspondences are taken as primitives.

imposes that plays belonging to the same information set of a player must have the same length. That is, players always know at which stage the game is. This condition is needed for the game (structure) of interest to be of the multistage kind (cf. Kuhn, 1953).

As already mentioned, information sets are hybrid concepts that may fail to adhere to the separation principle, as they may represent situations of genuine ignorance about the game unfolding, or cognitive failures in information retention, or both. The following one-player game illustrates a failure of memory.

Example 3 (Did I lock the door?) After leaving her home, Alice no longer remembers whether she locked the door or not. When she realizes this, she can either go back and check or not. The standard multistage game structure portraying this situation is as follows.

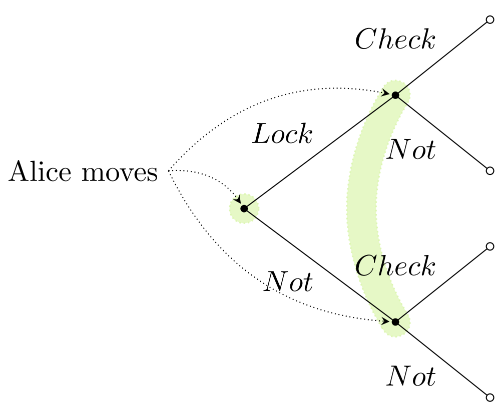

Figure 1 Alice does not remember if she locked the door.

In the graphical representation of game structures we are going to use shaded areas to represent information sets.

It is easy to check that all the conditions of Definition 2 are met. Specifically, (APF) is trivially satisfied because the set of players is a singleton. Moreover, (APF) holds because $\mathscr{Q}_{Alice} = \{\{(Lock), (Not)\}, \{\varnothing\}\}$, and $A_{Alice}(Lock) = \{Check, Not\} = A_{Alice}(Not)$. Lastly, $\ell((Lock)) = \ell((Not)) = 1$, and this verifies (KS).

Alice's failure in distinguishing plays $(Lock)$ and (Not) is a personal cognitive shortcoming made relevant by the fact that the rules of the game do not provide Alice with an automatic reminder of what she did. This clarifies the hybrid nature of information sets: in general, we have no guarantee at all that they represent a situation where players' ignorance of the game unfolding is induced by the rules of the game alone. ▲

To rule out situations where information sets encode some sorts of cognitive failures such as the one represented in Example 3, the notion of **perfect recall** has been proposed (cf., e.g., Kuhn, 1953, or Selten, 1975). As the name suggests, perfect recall rules out all the situations where information sets incorporate failures in players' ability to retain information.

The notion of perfect recall we are going to employ is formalized by means of the concept of "experience", as introduced by Osborne and Rubinstein (1994). As the game unfolds, a player goes through a sequence of information sets. Such sequence of information sets, coupled with the actions played at each such information set, forms the "experience" of a player within a play. To provide a formal definition of experience, we introduce the following notation. For each pair of plays $(v', v'') \in V \times V$ such that $v' \prec v''$ and for each player $i \in I$, we denote by $a_i(v', v'')$ the unique action such that $(v', (a_i(v', v''), a_{-i})) \preceq v''$ for some $a_{-i} \in A_{-i}$. In words, $a_i(v', v'')$ is the unique action of player i that does not prevent v'' from realizing when taken at v'. With this, the **experience function** X_i of player $i \in I$ can be defined recursively as follows. Fix a generic play $v \in V_i \cup \{\varnothing\}$.

- If $\ell(v) = 0$ (i.e., $v = \varnothing$), we define $X_i(\varnothing) := (Q_i(\varnothing)) = (\{\varnothing\})$.
- Assume that $X_i(u)$ has been defined for each $u \in V_i$ with $0 \leq \ell(u) \leq k$. Fix $v \in V_i$ with $\ell(v) = k+1$ (if any), and let $\mathrm{last}_i\, v$ be the longest predecessor of v where i is active (if any), or the empty play (otherwise). Note that $\mathrm{last}_i\, v \in V_i \cup \{\varnothing\}$ and $\ell(\mathrm{last}_i\, v) \leq k$. With this, we can define $X_i(v) := (X_i(\mathrm{last}_i\, v), a_i(\mathrm{last}_i\, v, v), Q_i(v))$.

The definition of perfect recall of Osborne and Rubinstein (1994) requires that, whenever two plays belong to the same information set of a player, then they induce the same experience for such player.

Definition 3 (Osborne & Rubinstein, 1994, Definition 203.3) *Fix a standard multistage game structure* $\Gamma_{SM} = \langle I, V, (A_i, \mathscr{Q}_i)_{i \in I} \rangle$; ***perfect recall** holds if:*

$$\forall\, i \in I, \forall\, v, w \in V_i, \quad \big(Q_i(v) = Q_i(w) \Longrightarrow X_i(v) = X_i(w)\big). \tag{PR}$$

A crucial feature of standard multistage game structures is that the information received and processed by players when they are not active *is not modeled*. This is in contrast with the flow-based approach presented above, where one message is received by each player after *each* stage. This observation motivates the "extension" of standard information structures in order to model the information players receive when they are inactive – this is necessary if we want to carry out a meaningful comparison of such stock-based information structures with flow-based ones.

4.2 Completions of standard information structures

We are now interested in "extending" standard information structures. The most intuitive way to do so consists in letting an information structure of a player be a partition of *all* the plays, even of those where such player is not active. We label the resulting information structures (as well as the corresponding game structures) as "synchronic", to convey the idea that all players simultaneously receive some information at each stage.[16]

Definition 4 *A* ***synchronic-information (SI) multistage game structure*** *is a tuple*

$$\Gamma_{SI} = \langle I, V, (A_i, \mathscr{R}_i)_{i\in I}\rangle,$$

where I, A_i, and V are as in Definition 2 and $\mathscr{R}_i$ is the ***SI information structure*** *of player i ($i \in I$), with $\mathscr{R}_i$ being a partition of V that satisfies the following properties:*

$$\forall\, v, w \in V, \quad \big(R_i(v) = R_i(w) \Longrightarrow A_i(v) = A_i(w)\big); \qquad \text{(KfA-SI)}$$
$$\forall\, v, w \in V, \quad \big(R_i(v) = R_i(w) \Longrightarrow \ell(v) = \ell(w)\big). \qquad \text{(KS-SI)}$$

Conditions (KfA-SI) and (KS-SI) are essentially the same as in Definition 2, and their labels remark the fact that they are defined for an SI information structure rather than for a standard one.

The notion of perfect recall is straightforwardly extended to the SI case. The definition of experience functions is adapted in a natural way: given that players receive some information after each stage, in the recursive definition of experience it is enough to focus on the immediate predecessor of a given play, rather than on the longest predecessor where a given player was active.

Definition 5 (Perfect recall, SI) *Fix an SI multistage game structure $\Gamma_{SI} = \langle I, V, (A_i, \mathscr{R}_i)_{i\in I}\rangle$.* ***Perfect recall*** *holds if:*

$$\forall\, i \in I, \forall\, v, w \in V, \quad \big(R_i(v) = R_i(w) \Longrightarrow X_i(v) = X_i(w)\big). \qquad \text{(PR-SI)}$$

[16] The terminology is borrowed from Battigalli and Bonanno (1999).

We can now analyze how a standard information structure can be turned into an SI one. As one may expect, one should first impose that the standard information structure and the SI one "agree", for each player, on how to partition the sets where such player is active. However, this is in general not enough, as the next example shows.

Example 4 (A quartet game) Assume that a game has to be played among Ann, Bob, Chloe, and Dave, so that $I = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$. As for potentially available actions, assume $A_{\mathrm{A}} = A_{\mathrm{B}} = \{\ell, r, w\}$, $A_{\mathrm{C}} = \{Ann, Bob, w\}$, $A_{\mathrm{D}} = \{u, d, w\}$, where w is the dummy action "wait".

The following is a graphical portray of a game structure, with Dave's (both standard and SI) information structure. For the sake of the argument, we can assume that all other players are perfectly informed – that is, that their information structures are made of singletons.

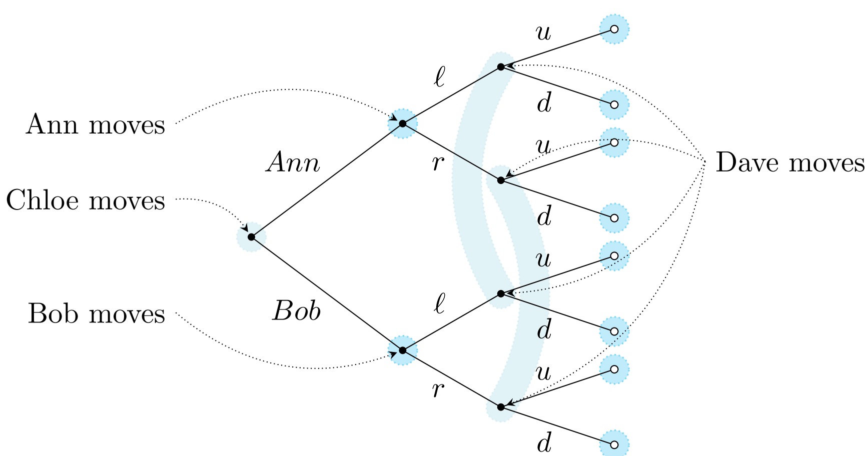

Figure 2 A game tree with Dave's standard (in light blue) and SI information structures (in light and darker blue): (PR) holds, but (PR-SI) fails.

In Figure 2, the two information structures intuitively "agree". Yet, it is easy to check that the standard information structure satisfies perfect recall, while the SI one does not. To appreciate it, note that (PR-SI) implies that whenever two plays belong to different information sets, their successors must also belong to different information sets. This obviously fails in Figure 2, as (Ann) and (Bob) belong to different information sets of Dave, but $(Ann, \ell) \succ (Ann)$ and $(Bob, \ell) \succ (Bob)$ belong to the same information set. In words, the SI information structure of Figure 2 describes a situation in which Dave observes Chloe's move, but later on he forgets it. ▲

Example 4 shows that something more than "agreement" is needed for perfect recall in a standard information structure to be preserved when we move to an SI setting. In the following, we are going to denote generic standard and SI information structures (obviously defined starting from the same tree V) as $(\mathscr{Q}_i)_{i\in I}$ and $(\mathscr{R}_i)_{i\in I}$, respectively. For $(\mathscr{R}_i)_{i\in I}$ to "reasonably complete" a standard information structure $(\mathscr{Q}_i)_{i\in I}$ *satisfying perfect recall*, some requirements have to be met.[17] First, $\mathscr{R}_i$ $(i \in I)$ must be *consistent* with $\mathscr{Q}_i$ in the way it partitions V_i. Second, information structure $(\mathscr{R}_i)_{i\in I}$ must be such that players *do not forget* the past: whenever w is a successor of v, then plays that can be reachable from w must also be reachable from v or from some play which is indistinguishable from v according to $(\mathscr{R}_i)_{i\in I}$ (and the same must apply to each player). Third, players must *recall their own actions*: for each player i, if play w comes immediately after play v (which belongs to information set $R_i(v)$) and player i took action a at v, then all plays in $R_i(w)$ must be such that player i took action a after some play in $R_i(v)$.

We need some preliminary notation. Let $V(R_i, a_i)$ denote the set of plays that immediately follow those in information set R_i, in which player i chooses a_i at R_i – that is, $V(R_i, a_i) := \{w \in V : \exists\, v \in R_i, \exists\, a_{-i} \in A_{-i}(v),\ (v, (a_i, a_{-i})) = w\}$.

Definition 6 *Fix a standard information structure* $(\mathscr{Q}_i)_{i\in I}$ satisfying perfect recall. *An SI information structure* $(\mathscr{R}_i)_{i\in I}$ *is a* ***completion*** *of* $(\mathscr{Q}_i)_{i\in I}$ *if it satisfies the following:*

$$\forall\, i \in I, \forall\, v \in V_i, \quad Q_i(v) = R_i(v); \tag{C}$$

$$\forall\, i \in I, \forall\, v, w \in V, \quad \big(v \prec w \Longrightarrow \forall\, w' \in R_i(w), \exists\, v' \in R_i(v), v' \prec w'\big); \tag{NF}$$

$$\forall\, i \in I, \forall\, v \in V, \forall\, a_i \in A_i(v),$$
$$\big(v \in V(R_i(\operatorname{pre} v), a_i) \Longrightarrow R_i(v) \subseteq V(R_i(\operatorname{pre} v), a_i)\big). \tag{ROA}$$

For each standard multistage game structure $\Gamma_{SM} = \langle I, V, (A_i, \mathscr{Q}_i)_{i\in I}\rangle$ with perfect recall, we call a **completion** of Γ_{SM} any SI multistage game structure $\Gamma_{SI} = \langle I, V, (A_i, \mathscr{R}_i)_{i\in I}\rangle$, such that $(\mathscr{R}_i)_{i\in I}$ is a completion of $(\mathscr{Q}_i)_{i\in I}$. Conversely, for each SI multistage game structure $\Gamma_{SI} = \langle I, V, (A_i, \mathscr{R}_i)_{i\in I}\rangle$, we let its **standard restriction** be the standard multistage game structure $\Gamma_{SM} = \langle I, V, (A_i, \mathscr{Q}_i)_{i\in I}\rangle$ where $(\mathscr{Q}_i)_{i\in I}$ is such that, for each $i \in I$ and $v \in V_i$, $Q_i(v) := R_i(v)$.

As one may expect, a completion is a way to "extend" a standard information structure that preserves perfect recall. Indeed, condition (C) ensures

[17]The following conditions are taken verbatim from Battigalli and Bonanno (1999).

"agreement" between the standard information structure and its completion. Then, the other conditions ensure that perfect recall holds in the derived setting, as the next result shows.[18]

Lemma 1 *Fix a standard multistage game structure* $\Gamma_{SM} = \langle I, V, (A_i, \mathscr{Q}_i)_{i\in I}\rangle$ *satisfying* (PR)*, and let* $\Gamma_{SI} = \langle I, V, (A_i, \mathscr{R}_i)_{i\in I}\rangle$ *be one of its completions. Then,* Γ_{SI} *satisfies* (PR-SI)*.*

For instance, the SI information structure of Figure 2 is not a completion because it allows Dave to forget about some previously known information – that is, he observes Chloe's action, but later on he forgets it. In a proper completion of such information structure, Dave must also be uncertain of Chloe's action in the first place (cf. Figure 3 below). Conversely, it is also easy to check that, starting from a SI multistage game structure with perfect recall, its standard restriction also satisfies perfect recall.

Now, it is legitimate to ask whether these descriptions of a game structure can be turned into "equivalent" representations using the flow-based approach.

5 Information flows and perfect recall

The aim of this section is to relate the two approaches described so far. Section 5.1 discusses how SI information structures can be derived by flow-based representations, under the interpretive assumption that players memorize and efficiently use the information they observe to make inferences. Section 5.2 establishes an equivalence result between our formalism and the conventional one.

5.1 Making inferences through messages

In this subsection, we describe the inferences a player can make using the flows of information determined by the rules of the game. In particular, we are going to show how a flow-based multistage game structure can be turned into an SI one under the interpretive assumptions that players know how feedback is generated and have perfect mnemonic abilities. To do this, we first restrict attention to *plays*, rather than to extended histories, consistently with the standard approach. Then, we show how information sets may be derived from the feedback functions.

[18] Proofs are collected in Appendix A.

Our approach is intuitive: as a player receives messages about how the play is unfolding, she is able to combine such stream of pieces of information with the actions she took to identify the set of plays that are consistent with this evidence. Such "set of indistinguishable plays" is an information set of the derived SI multistage game form: we then show that (PR-SI) is satisfied, that is, the derived collection of information sets is a partition satisfying the perfect recall property.

Fix a flow-based multistage game structure $\Gamma_{FM} = \langle I, (A_i, M_i, f_i, \mathcal{A}_i)_{i \in I} \rangle$. For each play $(a_k)_{k=1}^{t} \in P$, there is a unique extended history in H consistent with it, namely, $(m_0, a_1, f^1(a_1), a_2, f^2((a_1, a_2)), \ldots, a_t, f^t((a_1, \ldots, a_t)))$. For each play $p \in P$, we let $E(p)$ denote the extended history derived as above. Formally, $E : P \to H$ is the inverse of the projection $\mathrm{proj}_P : H \to P$ that maps each extended history to the corresponding play. One can check that E is a well-defined map because proj_P is a bijection.[19]

Note that two extended histories $g, h \in H$ cannot be distinguished by player i ($i \in I$) if they correspond to the same personal history (or cumulated information) of i, that is, if $C_i(g) = C_i(h)$. Thus, we say that two plays $p, q \in P$ are **indistinguishable** for player i ($i \in I$), written $p \sim_i q$, if the extended histories in H inducing them result in the same stream of information for i:

$$p \sim_i q \Longleftrightarrow C_i(E(p)) = C_i(E(q)),$$

where $E(p)$ is the unique extended history induced by play p. To ease notation, for each $i \in I$, we let $F_i := C_i \circ E : P \to H_i$ denote the map from plays to personal histories for i. It is easily verified that, for each $i \in I$, $\sim_i$ is an equivalence relation on P.[20] The following can be checked by inspection of the definition of $\sim_i$.

Remark 2 Fix a flow-based multistage game structure Γ_{FM}. For each $i \in I$ and $p, q \in P$, $p \sim_i q$ implies: (1) $\ell(p) = \ell(q)$, (2) $\mathrm{proj}_{A_i^{\leq T}} p = \mathrm{proj}_{A_i^{\leq T}} q$, (3) $\mathrm{pre}\, p \sim_i \mathrm{pre}\, q$, (4) $\mathcal{A}_i\left(f_i\left(p\right)\right) = \mathcal{A}_i\left(f_i\left(q\right)\right)$. ♦

[19] To check that proj_P is injective, consider $g, h \in H$ such that $g \neq h$. If the two extended histories have different lengths, then so will their projections, proving $\mathrm{proj}_P\, g \neq \mathrm{proj}_P\, h$. Assume then $\ell(g) = \ell(h)$ and proceed by induction. For the basis step, let $\ell(g) = \ell(h) = 1$, so that $g = (a, f(a))$ and $h = (b, f(b))$ for some $a, b \in A$. Obviously, $g \neq h$ if and only if $a \neq b$, in which case we also have $\mathrm{proj}_P\, g = a \neq b = \mathrm{proj}_P\, h$. The proof of the inductive step is analogous. Surjectivity of proj_P is obvious.

[20] We find it more convenient to state this condition in terms of plays rather than extended histories. In any case, the (finite) sets P and H are isomorphic, as mentioned in Section 3.

For each $p \in P$, we let $[p]_{\sim_i} := \{q \in P : p \sim_i q\}$ be the equivalence class of p. For each $i \in I$, the quotient space $P/\sim_i$, defined as $\mathscr{R}_i := \{R_i \subseteq P : \exists\, q \in P, R_i = [p]_{\sim_i}\}$, is a partition of plays according to player i's indistinguishability relation. It will represent the collection of information sets of player i in the SI multistage game structure derived from Γ_{FM}: indeed, when $p \in P$ realizes, player i observes the information stream $F_i(p)$ and she may deem possible all the plays that induce the same stream, that is, all the plays in $[p]_{\sim_i}$. The fact that P is a tree (Remark 1) and Remark 2 imply:

Remark 3 Structure $\langle I, P, (A_i, \mathscr{R}_i)_{i \in I}\rangle$ is an SI multistage game structure. ♦

With this, we let $\mathrm{SI}(\Gamma_{FM}) = \langle I, P, (A_i, \mathscr{R}_i)_{i \in I}\rangle$ denote the **SI multistage game structure derived from Γ_{FM}**.

Proposition 1 *For any flow-based multistage game structure Γ_{FM}, $\mathrm{SI}(\Gamma_{FM})$ satisfies the perfect recall property* (PR-SI)*.*

Proposition 1 gives an important intuition. We talked of information sets as representing *stocks* of information, or *snapshots* of what players deem possible as the game unfolds: if the realized play is p, then player i ($i \in I$) "knows" that the one of the plays in $R_i(p)$ has realized. But the formalism behind information sets is silent on *how* and *from what* such "knowledge" is derived. If we stick to the proposed flow-based representation, instead, not only can such snapshots be seamlessly derived, but it also becomes crystal clear that they reflect players' *accumulated* information as specified by the rules of interaction, in compliance with the separation principle.

It is worth noting that, in the current formulation, two distinct plays can be distinguished by player $i \in I$ if and only if they induce different realized streams of pieces of information (i.e., played actions and received messages). This means that one must keep track of previous evidence in order to assess indistinguishability of plays according to relations $(\sim_i)_{i \in I}$. To interpret this requirement, we can think of indistinguishability of plays (from some player's perspective) as something that can be assessed either by an external observer, or by a player that memorizes the pieces of evidence she observes throughout the game. This is obviously an informal and interpretive assumption, and a formal analysis would involve a mathematical description of players' ability to retain information. In Section 7 we are going to revisit Proposition 1 by introducing a formal description of players' memory.[21]

[21] We reserve the term "recall" to denote properties concerning the standard represen-

Lastly, it is interesting to observe that a possibly different standard multistage game structure could be obtained by making indistinguishability a different requirement. Indeed, we could consider a profile $(\approx_i)_{i \in I}$ of relations on P such that, for each $i \in I$ and $p, q \in P$, $p \approx_i q$ if and only if $f_i(p) = f_i(q)$: in such case, we say that p and q are **minimally indistinguishable**. That is, two plays are minimally indistinguishable for player $i \in I$ if and only if they result in the same message for player i: in such case, one no longer has to keep track of all the evidence observed, but only of the last message. Then, the standard multistage game structure obtained considering the quotient spaces $(P/\approx_i)_{i \in I}$ as the players' information structures satisfies perfect recall if and only if the rules of the game themselves remind to each player and at each stage all the information previously provided to such player. It is possible to check that perfect recall holds in the derived game if and only if the original flow-based multistage game features cumulative feedback.[22]

5.2 Characterizing perfect recall

We showed in Proposition 1 that a flow-based multistage game structure induces an SI multistage game structure that naturally satisfies perfect recall. In this section, we go a step further in investigating the link between perfect recall and flows of information. Specifically, we characterize perfect recall in SI multistage game structures (Proposition 2), and then we adapt such result to standard multistage game structures (Corollary 1) by exploiting the notion of completion introduced in Section 4.2.

tation of information sets, because such meaning is entrenched in the literature. Thus, we use the word "memory" whenever we (formally or informally) refer to players' ability to retain information in a flow-based setting. This terminological distinction is useful to remind of a relevant difference: while (perfect of imperfect) recall is a property of a standard game structure, memory is something that we *append* to a flow-based description of the rules of the game. To put it differently, unlike "recall", "memory" is a *personal feature* of players.

[22] Indeed, an implicit assumption of our flow-based approach is that players correctly perceive and remember the last message they observe, as such message is needed to figure out the feasible actions. Under cumulative feedback, the messages a player receives encode all the previous information available to such player, including her past actions. Hence, remembering the last message implies being able to retrieve all previous information. Therefore, a standard multistage game tree with perfect recall may be interpreted as either a game played by agents that correctly recall what they do and observe, or as a game where players are constantly reminded of the information they have been provided with. In any case, the conventional representation fails to specify how much of players' "knowledge" of the game unfolding may be ascribed to the rules of the game, as opposed to cognitive considerations.

Proposition 2 *Fix an SI multistage game structure* Γ_{SI}. (PR-SI) *holds in* Γ_{SI} *if and only if there exists a flow-based multistage game structure* Γ_{FM} *such that* $\Gamma_{SI} = \mathrm{SI}(\Gamma_{MF})$.

The intuition behind the proof of Proposition 2 is as follows. The "if" direction is given by Proposition 1. As for the "only if" direction, it is enough to construct a suitable flow-based multistage game structure Γ_{FM} that induces Γ_{SI}. To do so, one may simply define the feedback functions to be such that, after a play realizes, each player is told her information set in Γ_{SI} to which such play belongs. Action feasibility correspondences are instead directly retrieved from the game structure Γ_{SI}, noting that the messages players observe are actually information sets of Γ_{SI}, which therefore also inform players of their feasible actions as per condition (KfA). In this way, the sets of feasible histories of Γ_{SI} and $\mathrm{SI}(\Gamma_{FM})$ are the same. With such construction, for each player, two plays are indistinguishable if and only if they induce the same experience: this, together with perfect recall, ensures that the information structure Γ_{SI} coincides with that of $\mathrm{SI}(\Gamma_{FM})$.

While Proposition 1 established that each flow-based multistage game structure induces an SI multistage game structure satisfying perfect recall, Proposition 2 adds the other direction: any SI multistage game structure with perfect recall is induced by some flow-based multistage game structure. All in all, it is possible to claim that, for an SI information structure, satisfying perfect recall is equivalent to being obtainable from some suitable structure with information flows under the maintained interpretive assumption that players have perfect mnemonic abilities. We are going to make such assumption precise in Section 7.

We now turn to standard multistage game structures. As expected, the concept of completions of standard information structures makes the transition smooth. Combining Proposition 2 and Lemma 1, the following obtains.

Corollary 1 *The following are true:*

1. *Each of the completions of a standard multistage game structure satisfying* (PR) *is induced by some flow-based multistage game structure.*

2. *Each flow-based multistage game structure induces the completion of some standard multistage game structure satisfying* (PR).

We illustrate the foregoing results by means of an example.

Example 5 (A quartet game, continued) The two possible completions of Dave's standard information structure of Figure 2 are in Figure 3. With either of them,[23] (PR-SI) holds as per Lemma 1 (point 1).

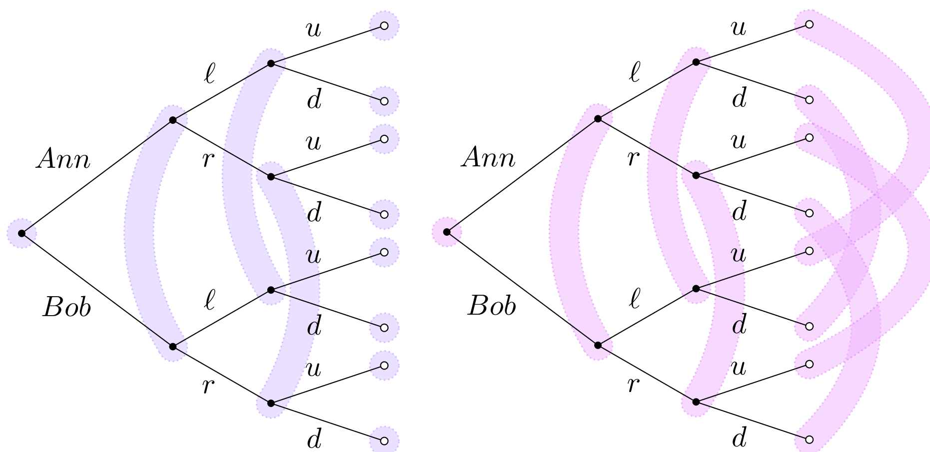

Figure 3 Two completions of Dave's standard information structure of Figure 2: (PR-SI) holds.

As per Corollary 1 (point 1), each of the information structures of Figure 3 can be induced by a suitable stream of pieces of information. Table 1 reports the key features of M_{D}, f_{D}, and $\mathcal{A}_{\mathrm{D}}$ of the flow-based multistage game structure mimicking the information structure on the left of Figure 3.

Play	**Message**	**Feasible actions**
$\varnothing_A$	*The game started.*	$\{w\}$
$(Ann), (Bob)$	*Chloe moved.*	$\{w\}$
$(Ann, \ell), (Bob, \ell)$	*Someone played ℓ.*	$\{u, d\}$
$(Ann, r), (Bob, r)$	*Someone played r.*	$\{u, d\}$
$(Ann, \ell, u), (Ann, \ell, d)$, $(Ann, r, u), (Ann, r, d)$	*Ann played.*	$\emptyset$
$(Bob, \ell, u), (Bob, \ell, d)$, $(Bob, r, u), (Bob, r, d)$	*Bob played.*	$\emptyset$

[23]Recall that, for the sake of simplicity, we assume that the other players are perfectly informed about the game unfolding.

Table 1 Information flow inducing Dave's information structure of Figure 3 (left).

Note that the last message Dave can get according to the information flows described in Table 1 does not *per se* allow him to understand which terminal play realized. To figure this out, it has to be combined with the previous message. Such inference is allowed by the rules of the game, and it is possible under the informal assumption of perfect memory (see the discussion about indistinguishability of plays in Section 5.1).

We easily obtain an analogous table for the information structure on the right of Figure 3. The only difference is that messages are less informative in such case, as the messages Dave observe after acting do not shed light on who moved before him. Hence, information is in some sense coarser here.

Play	**Message**	**Feasible actions**
$\varnothing_A$	*The game started.*	$\{w\}$
$(Ann), (Bob)$	*Chloe moved.*	$\{w\}$
$(Ann, \ell), (Bob, \ell)$	*Someone played ℓ.*	$\{u, d\}$
$(Ann, r), (Bob, r)$	*Someone played r.*	$\{u, d\}$
$(Ann, \ell, u), (Ann, \ell, d)$ $(Ann, r, u), (Ann, r, d)$ $(Bob, \ell, u), (Bob, \ell, d)$ $(Bob, r, u), (Bob, r, d)$	*Game over!*	$\emptyset$

Table 2 Information flow inducing Dave's information structure of Figure 3 (right).

Point 2 of Corollary 1 is immediate once we consider the standard multistage game structure which is PM information equivalent to any of the information flow specifications of Tables 1 and 2: such standard multistage game structures feature the information structures depicted in Figure 3, which clearly are completions of their standard restrictions as per point 2 of Lemma 1. ▲

6 Generalizing the framework

Games with a multistage structure form a rich and widely studied class, and our approach is well-suited to analyze them. In this section we describe how to extend the analysis to relax the multistage assumption. An example may be useful to build the intuition.

Example 6 (Selten's horse) The following (standard) game structure is well-known. Figure 4 portrays a game structure and Chloe's information sets.

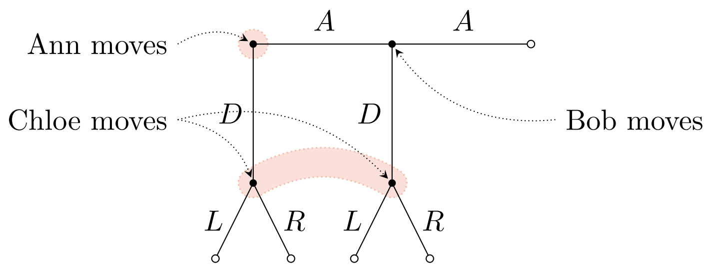

Figure 4 Selten's horse with Chloe's information sets.

It is easy to verify that condition (KS) of Definition 2 fails here, as the plays in information set $\{(\mathrm{D}), (\mathrm{A}, \mathrm{D})\}$ have different lengths. In particular, when she has to move, Chloe does not know whether both her co-players already acted or Ann gave her the move right away. ▲

Our flow-based approach as specified in Section 3 fails to give an account of the information structure of Figure 4. This is due to the fact that so far we assumed that a message profile is generated immediately after a move is taken. However, it is possible to overcome such limitation in a natural way, as the inspection of Example 6 reveals. Specifically, we need to allow for the possibility that no message be received by some players after some plays. For instance, in Example 6, Chloe should not observe any message after (A). Indeed, suppose she were told "Ann moved, now you can only wait." Then at her information set she would have to forget that she waited.[24]

Next, we first describe how to adapt our flow-based approach to non-multistage games (Section 6.1), then we generalize the standard approach (Section 6.2), and we conclude by extending the results of Section 5 to the non-multistage case (Section 6.3).

[24]The only way to transform the "horse" of Figure 4 into a multistage game where players are always "alert" is to introduce a dummy node after (D) and have plays (D) and (A) generate the same message for Chloe saying that Ann moved and she can only wait. If Bob does not go across (because he does not move, or he goes down), then Chloe is told "it is your turn, you can go left or right." According to traditional game theory, the game tree of Figure 4 and the multistage game tree resulting from this transformation are "equivalent." Yet, it can be shown by example that some non-multistage game trees cannot be turned into multistage ones (see Battigalli & Bonanno, 1999).

6.1 Flows

This subsection is similar to Section 3, and it discusses the key ingredients for the definition of a flow-based game structure.

Players, messages, and actions The sets I, $(A_i)_{i\in I}$ are as before, the message sets $(M_i)_{i\in I}$ are posited as primitive objects. In this more general setting, messages serve an additional purpose. Unlike in Section 3, players are not assumed to be always "alert" – see Example 6, where Chloe does not process any information immediately after Ann chooses A. Therefore, we can think of messages as ways to alert players – that is, they inform players that something happened in the meantime.

Moreover, not all players necessarily move or receive messages at the same time: for instance, in Example 6 some message needs to be generated after Ann's action A to inform Bob that he can move, but Chloe should definitely *not* observe any message at that point. As a result, it is convenient to define, for each nonempty $J \subseteq I$, the sets $A_J := \bigtimes_{i\in J} A_i$ and $M_J := \bigtimes_{i\in J} M_i$ of profiles of actions and messages of players in subset J. Changing our multistage notation, here we let $A := \bigcup_{\emptyset\neq J\subseteq I} A_J$ and $M := \bigcup_{\emptyset\neq J\subseteq I} M_J$. In words, $a \in A$ is a profile of actions $(a_i)_{i\in J}$ for an unspecified and nonempty $J \subseteq I$, and an analogous interpretation applies to elements of M. Thus, from this section onward, A and M *no longer* stand for $\bigtimes_{i\in I} A_i$ and $\bigtimes_{i\in I} M_i$, but rather for sets of profiles of actions or messages for some nonempty subset of players.

Formally, profiles are functions that associate each player in a subset $J \subseteq I$ to a corresponding object, such as an action or a message. With this, it is useful to define the correspondence $\mathcal{D} : M \cup A \rightrightarrows I$ to be such that, for each $b \in A \cup M$, $\mathcal{D}(b)$ is the *domain* of profile (function) b. As a matter of terminology, we say that player $i \in I$ is **alert** given message profile $m \in M$ if $i \in \mathcal{D}(m)$. Equivalently, m alerts player i. The interpretation is straightforward: alert players are the ones who receive information when a message profile is generated.

A rule to determine feasible actions As discussed in Section 3, we assume that players understand the action they can take by looking at the last message they receive as the play unfols. Hence, for each player $i \in I$, we posit an **action feasibility correspondence** $\mathcal{A}_i : M_i \rightrightarrows A_i$ as in Section 3.

Our formalism then implies that being alert is a prerequisite for acting – that is, players can move only if they receive some message, and hence only if they are alerted by some profile $m \in M$. Hence, we say that player $i \in I$

is **active** after $m \in M$ if $i \in \mathcal{D}(m)$ and $|\mathcal{A}_i(\text{proj}_{M_i} m)| > 1$. As already mentioned, moves are taken after receiving information, so a player is active after some message profile if (i) she received some information, and if (ii) she can choose from at least two actions based on the piece of information she received. Note that point (ii) allows an alert player to have only one feasible action – in such case, a player is alert but inactive, meaning that she receives and processes information without acting.

Lastly, it is convenient to define the joint feasibility correspondence $\mathcal{A} : M \rightrightarrows A$, as $m \mapsto \bigtimes_{i\in\mathcal{D}(m)} \mathcal{A}_i(\text{proj}_{M_i} m)$. Thus, $\mathcal{A}(m)$ ($m \in M$) is the set of action profiles that may be taken (by alert players) after m.

A rule to generate messages We represent how messages are generated by means of a **collective feedback function** $\tilde{f} : A^{\leq T} \to M$. Note that $A^{\leq T}$ is a potentially large set, but combining the feedback function with the action feasibility correspondences, one gets the subset of $A^{\leq T}$, which contains all the *feasible* sequences of action profiles. So, we are ultimately interested in considering the restriction of $\tilde{f}$ to such set. Yet, such set of feasible plays is a *derived object* – that is, it needs to be retrieved in a recursive way by exploiting the action feasibility correspondences $(\mathcal{A}_i)_{i\in I}$ and the feedback function $\tilde{f}$.

It is convenient to refer to a player's **individual feedback function**, which directly specifies the message a given player $i \in I$ would observe after $a^t \in A^{\leq T}$ (if any). For each player $i \in I$, let $\tilde{f}_i : \text{dom}\,\tilde{f}_i \to M_i$ be defined as follows. First of all, $\text{dom}\,\tilde{f}_i := \{a^t \in A^{\leq T} : i \in \mathcal{D}(\tilde{f}(a^t))\}$. In words, the domain of $\tilde{f}_i$ is the set of sequences of action profiles generating a message that alerts player i: this is clearly motivated by the observation that player i does not necessarily receive a message after an arbitrary sequence $a^t \in A^{\leq T}$. Then, for each $a^t \in \text{dom}\,\tilde{f}_i$, let $\tilde{f}_i(a^t) := \text{proj}_{M_i} \tilde{f}(a^t)$.

Some restrictions We can now combine feasibility correspondences and message generating functions in order to see how they shape the game unfolding. Before doing so, however, we have to impose some natural restrictions that apply to $(\mathcal{A}_i)_{i\in I}$ and $\tilde{f}$.

- *Know that the game started.*It is reasonable to assume that the first message profile to ever be generated alerts everyone. Obviously, not all players need to be active afterwards, but all players should be informed that the game started. In this regard, the first message player $i \in I$ receives may be thought of as stating "the game is started" and specifying i's initially feasible actions. Formally, we impose:

$$\tilde{f}(\varnothing_A) \in M_I. \tag{KGS-F}$$

- *Know that the game ended.* The game ends after $m \in M$ if $\mathcal{A}(m) = \emptyset$. In such case, we say that m **terminates the game**. Specifically, given that $\mathcal{A}(m) = \bigtimes_{i \in \mathcal{D}(m)} \mathcal{A}_i(\operatorname{proj}_{M_i} m)$ for each $m \in M$, we require that whenever $\mathcal{A}_j(\operatorname{proj}_{M_i} m) = \emptyset$ for some $j \in \mathcal{D}(m)$, then $\mathcal{A}_i(\operatorname{proj}_{M_i} m) = \emptyset$ for all $i \in \mathcal{D}(m)$. Therefore, as soon as the game is over for some player (i.e., such player does not have any feasible actions anymore) it is over for everyone. Just like it was reasonable to require that all players be informed of the game start, it is equally compelling to require that all players be informed of the game end. That is, whenever a message profile terminates the game, then it must alert everyone. Furthermore, since the maximal duration of the game is T, the game must end after each sequence of action profiles of length T. Formally:

$$\begin{aligned} \forall\, m \in M, &\quad \mathcal{A}(m) = \emptyset \Longrightarrow m \in M_I, \\ \forall\, a^T \in A^T, &\quad \mathcal{A}\left(\tilde{f}(a^T)\right) = \emptyset. \end{aligned} \tag{KGE-F}$$

We can now give the definition of game structure.

Definition 7 *A **flow-based game structure** is a tuple*

$$\Gamma_F = \langle I, \tilde{f}, (A_i, M_i, \mathcal{A}_i)_{i \in I} \rangle,$$

where the elements are as above, and (KGS-F) *and* (KGE-F) *are satisfied.*

Possible plays Our objective is now to derive all the possible ways in which the game may unfold given a structure $\Gamma_F = \langle I, \tilde{f}, (A_i, M_i, \mathcal{A}_i)_{i \in I} \rangle$. In the following, we maintain the usual distinction between (partial or complete) **plays** and **extended histories**. We will also define a feedback function $f : P \to M$, which will be obtained by restricting the collective feedback function $\tilde{f} : A^{\leq T} \to M$ to the set of feasible sequences of action profiles (i.e., to the set of plays).

The starting point is obvious: define the sets of length-0 plays and extended histories as $P^0 := \{\varnothing\}$ and $H^0 := \{\varnothing\}$.[25] Then, let $f^0 : P^0 \to M$ be such that $f^0(\varnothing) := \tilde{f}^1(\varnothing)$.

[25] Recall that we drop the range subscript from the empty sequence symbol when no confusion may arise; e.g., we write just $\varnothing$ instead of $\varnothing_A$ as the unique element of A^0.

Given P^0, H^0, and f^0, we can obtain the set of length-1 plays and histories as

$$P^1 := \mathcal{A}(f^0(\varnothing)), \quad H^1 := \{f^0(\varnothing)\} \times \left(\bigcup_{a^1 \in P^1} \{a^1\} \times \{f^1(a^1)\} \right),$$

where, $f^1 : P^1 \to M$ is obtained by restricting $\tilde{f}$ to P^1. Lastly, the set of length-1 terminal extended histories is[26]

$$Z^1 := \{(m_0, a_1, m_1) \in H^1 : \mathcal{A}(m_1) = \emptyset\}.$$

Note that terminal histories feature as last element a message profile with domain I that terminates the game, and this verifies condition (KGE-F).

Assume now that P^t and H^t have been defined for $0 < t < T$. Then,

$$\begin{aligned}
P^{t+1} &:= \bigcup_{a^t \in P^t} \{a^t\} \times \mathcal{A}(f^t(a^t)), \\
H^{t+1} &:= \bigcup_{h \in H^t} \{h\} \times \left(\bigcup_{a \in \mathcal{A}(f^t(\mathrm{proj}_{P^t} h))} \{a\} \times \left\{ f^{t+1}\big((\mathrm{proj}_{P^t} h, a)\big) \right\} \right), \\
Z^{t+1} &:= \{(m_0, \dots, a_{t+1}, m_{t+1}) \in H^{t+1} : \mathcal{A}(m_{t+1}) = \emptyset\},
\end{aligned}$$

where f^t and f^{t+1} are the restrictions of $\tilde{f}$ to P^t and P^{t+1}, respectively.

Wrapping up, we let $P := \bigcup_{t=0}^T P^t$ be the set of feasible **plays**, $H := \bigcup_{t=0}^T H^t$ the set of feasible **extended histories**, and $Z := \bigcup_{t=1}^T Z^t$ the set of feasible **terminal extended histories**. Let $f : P \to M$ denote the map $a^t \mapsto f^t(a^t)$. A **play** $p \in P$ is **terminal** if $f(p)$ terminates the game (or, equivalently, if it is induced by a terminal extended history). Lastly, player $i \in I$ is **alert after play** $p \in P$ if $i \in \mathcal{D}(f(p))$. We let P_i $(i \in I)$ denote the set of plays after which player i is alert.

As before, during the game players only observe the actions they play and the messages they receive. It is therefore important to "extract" from each extended history in H the pieces of information each player gets exposed in such history. To this end, for each player $i \in I$, we let $c_i : M \cup A \to M_i \cup A_i$ be defined, for each $b \in M \cup A$, as

$$c_i(b) := \begin{cases} \mathrm{proj}_{M_i \cup A_i} b & \text{if } i \in \mathcal{D}(b); \\ \varnothing_{M_i \cup A_i} & \text{otherwise.} \end{cases}$$

In words, $c_i(\cdot)$ $(i \in I)$ isolates i's component (if any) from a profile of messages or actions. We use it to obtain a straightforward adaptation

[26] The length of extended histories is understood as the length of the induced play.

of the cumulated information function introduced in Section 3, by letting $C_i : H \to \{m_0\} \times (A_i \times M_i)^{\leq T}$ denote the map $(m_0, (a_k, m_k)_{k=1}^t) \mapsto (m_{i,0}, (c_i(a_k), c_i(m_k))_{k=1}^t)$.[27]

With this, we define the set of **personal histories** of player $i \in I$ as $H_i := C_i(H)$. The interpretation is as in Section 3: $C_i(h)$ ($i \in I$, $h \in H$) is the personal history of i induced by h, that is, the cumulated information i is provided with given extended history h. Differently from Section 3, $C_i(h)$ and h need not have the same length, because i may not be alert given h or some prefix of h.

Example 7 (Selten's horse, continued) Chloe's information structure as depicted in Figure 4 can be induced by the following (partial) flow of information. To generate the set of feasible plays, it is enough to specify suitable feasibility correspondences for Ann and Bob.

Play	**Message**	**Available actions**
$\varnothing$	*The game is starting, please wait*	$\{wait\}$
$(\mathrm{D}), (\mathrm{A}, \mathrm{D})$	*It's your turn, choose L or R*	$\{L, R\}$
(A)	No message	Not alert

Such flows are enough to induce the standard information sets of Figure 4. Yet, we said that such flows of information are partial because we obviously need also to specify terminal information (i.e., which messages are generated after terminal plays). Since players' terminal information is neglected in the standard approach, our partial flow is nonetheless enough to obtain the desired information structure. ▲

Lastly, it is instructive to compare the present approach with the one of Section 3. The multistage-case formalism with synchronic information is retrieved by requiring that the range of function $\tilde{f}$ be included in M_I.[28] This amounts to assuming that each play induces a message profile alerting everyone – hence, each player receives some message after each move (or profile of simultaneous moves), and this allows to recreate the SI multistage structure.

[27] Note that for a generic set X and for each sequence ξ of elements of X, the following holds: $(\xi, \varnothing_X) = (\varnothing_X, \xi) = \xi$. This is why we can neglect any $\varnothing_{M_i \cup A_i}$ we may have in the sequence $(C_i(b_k)_{k=1}^t)$.

[28] And distinguishing messages received in different stages.

6.2 Stocks

The definition of a general standard game structure is very similar to the one provided in Section 4. Of course, some notational adjustments need to be made, and condition (KS) needs to be relaxed, as one may see by considering the information structure depicted in Figure 4. For each tree $V \subseteq A^{\leq T}$ and $v \in V$, let $A(v) := \{a \in A : (v, a) \in V\}$ denote the set of action profiles that are feasible after v.

Definition 8 *A* ***standard game structure*** *is a tuple*

$$\Gamma_S = \langle I, V, \mathcal{I}, (A_i, \mathscr{U}_i)_{i \in I} \rangle,$$

where

- *I is the set of* ***players*** *and, for each $i \in I$, A_i is the set of actions potentially available to i;*
- *$V \subseteq A^{\leq T}$ is a tree (call it "the* ***tree****", and its elements "****plays****");*
- *$\mathcal{I} : V \rightrightarrows I$ is the* ***alert-player correspondence****, and it satisfies the following properties:*[29]

$$\mathcal{I}(\varnothing) = I; \tag{KGS-S}$$

$$\forall\, v = (\operatorname{pre} v, a) \in V, \quad \mathcal{I}(\operatorname{pre} v) \neq \emptyset,\ a \in A_{\mathcal{I}(\operatorname{pre} v)}; \tag{APM}$$

$$\forall\, v \in V, \quad A(v) = \mathop{\times}_{i \in \mathcal{I}(v)} \operatorname{proj}_{A_i} A(v); \tag{APF-G}$$

- *for each $i \in I$, $\mathscr{U}_i$ is the collection of* ***information sets*** *of player i, and it is a partition of $V_i := \{v \in V : i \in \mathcal{I}(v)\}$ satisfying*[30]

$$\forall\, v, w \in V_i, \quad \big(U_i(v) = U_i(w) \Longrightarrow \operatorname{proj}_{A_i} A(v) = \operatorname{proj}_{A_i} A(w)\big). \tag{KfA-G}$$

Some comments are in order. First, the requirements imposed on the alert-player correspondence formalize restrictions that in some sense mimic the ones we imposed in our flow-based approach. Specifically, condition (KGS-S) (know that the game started) is the counterpart of (KGS-F) in a standard setting, and it imposes that everyone must be alert at the root.

[29] Conditions (APF-G) and (KfA-G) are obviously analogous to (APF) and (KfA). The letter "G" clarifies that they refer to the general, rather than to the multistage case.

[30] For each play $v \in V_i$, we denote as $U_i(v)$ the unique element of $\mathscr{U}_i$ containing v.

Condition (APM) (alert players move) also mirrors the assumption we made in Section 6.1 that being alert is a prerequisite for being active. Condition (APF-G) (action profiles feasibility) and (KfA) (knowledge of feasible actions) are analogous to the similar conditions of Section 4.

Second, as a matter of terminology, we say that player $i \in I$ is **alert** at $v \in V$ if $i \in \mathcal{I}(v)$, and that she is **active** at $v \in V$ if $i \in \mathcal{I}(v)$ and $|\operatorname{proj}_{A_i} A(v)| > 1$. The set of plays where player $i \in I$ is alert is V_i, as already defined. Note that under condition (APM) being active implies being alert, so that decoupling "being alert" from "being active" allows us to represent situations where not only active players receive and process information.

The notion of experience function goes through almost verbatim as that of Section 4.1. In the present setting, however $\operatorname{last}_i v$ ($i \in I$, $v \in V_i$) denotes the longest strict predecessor of v where player i is alert.[31] Perfect recall is then easily defined.

Definition 9 *Standard game structure* $\Gamma_S = \langle I, V, \mathcal{I}, (A_i, \mathscr{U}_i)_{i \in I}\rangle$ *satisfies* ***perfect recall*** *if:*

$$\forall\, i \in I, \forall\, v, w \in V_i, \quad \big(U_i(v) = U_i(w) \Longrightarrow X_i(v) = X_i(w)\big). \tag{PR}$$

6.3 Revisiting previous results

As one may expect, we can retrieve from each flow-based game structure Γ_F a standard game structure $\operatorname{Std}(\Gamma_F)$, just like an SI multistage game structure was derived from a flow-based one in Section 5.1. We start from a flow-based game structure $\Gamma_F = \langle I, \tilde{f}, (A_i, M_i, \mathcal{A}_i)_{i \in I}\rangle$, and we consider the derived objects H, P, and f. The mathematical objects forming the game structure structure $\Gamma_S = \langle I, V, \mathcal{I}, (A_i, \mathscr{U}_i)_{i \in I}\rangle$ are derived as follows.[32]

- *Tree* (V). Let $V = P$.
- *Alert-player correspondence* ($\mathcal{I}$). Let $\mathcal{I}(v) = \mathcal{D}(f(v))$ for each $v \in V$. The interpretation is straightforward: after $v \in V = P$, the message profile $f(v)$ is generated, and players who are alerted by $f(v)$ are those in $\mathcal{D}(f(v))$. Hence, the alert players at v are the ones that are alerted by the message profile generated immediately after v.

[31] Formally, $\operatorname{last}_i v$ is the $\prec$-maximal element of $\{u \in V_i : u \prec v\}$.

[32] Throughout, we may assume that the set of players I and the set of feasible actions $(A_i)_{i \in I}$ are held fixed.

- *Information sets* $((\mathscr{U}_i)_{i\in I})$. For each player $i \in I$, we define the indistinguishability relation $\sim_i$ as we did in Section 5.1. In particular, we are going to make use again of the functions $(F_i)_{i\in I}$ that were defined in Section 5.1.[33] In this setting, however, the indistinguishability relation is defined only between plays after which player i is alert. Hence, we have to first retrieve the set of plays where i is alert, $V_i := \{v \in V : i \in \mathcal{I}(v)\} = \{v \in V : i \in \mathcal{D}(f(v))\}$. Then, for each $v, w \in V_i$, $v \sim_i w$ if and only if $F_i(v) = F_i(w)$, and we define $\mathscr{U}_i := \{U \subseteq V_i : \exists\, v \in V_i,\ U = [\,v\,]_{\sim_i}\}$.

It can be verified that all the requirements of a standard game structure spelled out in Definition 8 are satisfied by construction. With this, for each flow-based game structure Γ_F, we let $\mathrm{Std}(\Gamma_F) = \langle I, V, \mathcal{I}, (A_i, \mathscr{U}_i)_{i\in I}\rangle$ denote the standard structure whose elements are derived from Γ_F in the way we just described.

The following results adapt Propositions 1 and 2 to the general setting examined in the current section.

Proposition 3 *For each flow-based game structure* Γ_F*,* $\mathrm{Std}(\Gamma_F)$ *satisfies the perfect recall property* (PR)*.*

Proposition 4 *Fix a standard game structure* Γ_S*.* (PR) *holds in* Γ_S *if and only if there is a flow-based game structure* Γ_F *such that* $\mathrm{Std}(\Gamma_F) = \Gamma_S$*.*

The intuition behind the last results is very similar to that underpinning their multistage-case versions given the appropriate adjustments to allow players to be alert only at some points of the game. The formal proofs are in Appendix A.

7 Memory

In this section, we present a very basic and unstructured way to model individuals' ability to memorize information, which is *independent* from any specific context or game. As already stressed, our flow-based approach allows to cleanly and explicitly isolate players' game-specific information as implied by the rules of interaction, in adherence to the separation principle. Therefore, a rigorous and expressive language can be derived by combining

[33]Recall that, for each $i \in I$, $F_i : P \to H_i$ is the function mapping each play to the personal history of player i it induces. In the general setting, however, $F_i(p)$ ($p \in P$, $i \in I$) need not have the same length of p, as player i need not be always alert.

these two accounts of objectively provided and subjectively retained information. In particular, appending a description of players' cognitive features to a flow-based game structure allows to embed a wide array of cognitive limitations into specific interactive situations, and it allows to analyze them in a rigorous way. We believe this to be a key step towards the meaningful integration of cognitive failures into game-theoretic analyses.

To keep our analysis as general as possible, we are going to model memory using possibility correspondences. Hence, we are first going to presents basic definitions about such objects (Section 7.1). Then, we give a game-independent account of memory (Section 7.2). Lastly, we embed such analysis in a game-theoretic framework (Section 7.3).

7.1 Possibility correspondences

For an abstract set X, a **possibility correspondence** is any correspondence $\mathcal{P} : X \rightrightarrows X$. The conventional interpretation, which justifies the terminology used, is that X is a set of states, and $\mathcal{P}(x)$ ($x \in X$) is the set of states that are deemed possible by an agent when the true state is x (cf. Chapter 5 of Osborne & Rubinstein, 1994).

From a purely mathematical point of view, possibility correspondences can be thought of as ways to express binary relations over some set. In particular, consider a possibility correspondence $\mathcal{P} : X \rightrightarrows X$ and $x, y \in X$: if $y \in \mathcal{P}(x)$, then y is in relation with x. Therefore, it is natural to adapt properties of binary relations to possibility correspondences.

Definition 10 *Consider a possibility correspondence $\mathcal{P} : X \rightrightarrows X$.*

1. *$\mathcal{P}$ is* ***serial*** *if*
$$\forall\, x \in X, \quad \mathcal{P}(x) \neq \emptyset.$$
2. *$\mathcal{P}$ is* ***reflexive*** *if*
$$\forall\, x \in X, \quad x \in \mathcal{P}(x).$$
3. *$\mathcal{P}$ is* ***symmetric*** *if*
$$\forall\, x, y \in X, \quad \big(x \in \mathcal{P}(y) \Longrightarrow y \in \mathcal{P}(x)\big).$$
4. *$\mathcal{P}$ is* ***transitive*** *if*
$$\forall\, x, y, z \in X, \quad \big(x \in \mathcal{P}(y), y \in \mathcal{P}(z) \Longrightarrow z \in \mathcal{P}(x)\big).$$
5. *$\mathcal{P}$ is* ***partitional*** *if it is reflexive, symmetric, and transitive.*

Under the aforementioned interpretation of possibility correspondences, the names of the properties listed in Definition 10 are familiar. Seriality means that, at each state, some state is deemed possible. The other features are standard properties of binary relations, and their meaning is intuitive.

7.2 A game-independent analysis of memory

Consider a generic set X, and the set of sequences of elements of X of length equal to $N \in \mathbb{N}$ at most, $X^{\leq N}$. To ease the comparison with previous sections, X should be interpreted as a universal set of actions and messages – that is, $X = A^\star \cup M^\star$, where $A^\star$ and $M^\star$ are universal sets of actions and of messages, respectively.[34] Elements of $X^{\leq N}$ instead represent sequences of pieces of information an agent can observe. A **memory correspondence**[35] for agent i is a serial possibility correspondence $\mathcal{M}_i : X^{\leq N} \rightrightarrows X^{\leq N}$. In words, after observing sequence $\xi \in X^{\leq N}$, $\mathcal{M}_i(\xi) \subseteq X^{\leq N}$ is the set of sequences that are consistent with what agent i can recall of the actual sequence ξ. Hence, a correspondence $\mathcal{M}_i$ is ultimately a description of how precisely agent i stores and retains the information she receives.[36]

The most natural specification is the following.

Example 8 (Perfect memory) Agent i satisfies *perfect memory* if $\mathcal{M}_i$ is the correspondence $\xi \mapsto \{\xi\}$. ▲

This way of formalizing an agent's cognitive abilities allows to flexibly relax the assumption of perfect retention of information in interesting ways. For instance, allowing for bounded memory may be interesting in many economic applications involving interactions that are repeated many times.

[34] With the term "universal" we want to stress the idea that these are sets of conceivable actions the agent can take or of messages the agent can imagine to receive, and such actions and messages are not tied to any specific context or situation. Obviously, when a description of an agent's memory is embedded in a game tree, the set of actions she can take and of messages she can receive will be determined by the rules of interaction.

[35] Recall that we use the term "memory" when we consider an agent's personal cognitive feature, and we keep "(perfect) recall" to mean a property of information partitions in standard game structures.

[36] To be precise, memory can be though of as a two-step process involving both the *storage* and the subsequent *retrieval* of information. The latter process may well depend on the environmental cues an individual may observe: for instance, a cue may facilitate the retrieval of similar past experiences, while inhibiting that of less similar ones (see, e.g., Bordalo, Gennaioli, & Shleifer, 2020). In this setting, we are focusing on the first channel – that is, on how information is stored by agents. However, the analysis of the current section is abstract enough that it can easily be enriched in order to model phenomena involving cued recall of stored information.

In such cases, it may be reasonable to assume that players fail to keep track of all the outcomes of previous interactions.

Example 9 (Bounded memory) Agent i satisfies *k-bounded memory* ($k \leq N$) if $\mathcal{M}_i$ is the correspondence

$$(x_\ell)_{\ell=1}^n \mapsto \big\{(y_\ell)_{\ell=1}^n \in X^{\leq N} : \forall\, j \in \{0, \ldots, \min\{k, n\}\},\ y_{n-j} = x_{n-j}\big\}.$$

For an illustration, consider $\{0,1\}^{\leq 3}$, and assume agent i exhibits 1-bounded memory. Then, $\mathcal{M}_i\big((0,0,1)\big) = \big\{(1,1,1),(1,0,1),(0,1,1),(0,0,1)\big\}$.

Note that this definition assumes that agent i can recall the number of pieces of information she observed. That is, all the sequences in $\mathcal{M}_i(\xi)$ ($\xi \in X^{\leq N}$) must have the same length of ξ. We may want to allow agent i to forget such detail. In that case, we let $\mathcal{M}_i$ be

$$(x_\ell)_{\ell=1}^n \mapsto \big\{(y_\ell)_{\ell=1}^m \in X^{\leq N} : \forall\, j \in \{0, \ldots, \min\{k, m, n\}\},\ y_{m-j} = x_{n-j}\big\}.$$

For example, considering $\{0,1\}^{\leq 3}$ and 1-bounded recall, we obtain $\mathcal{M}_i\big((0,0,1)\big) = \big\{(1),(0,1),(1,1),(1,1,1),(1,0,1),(0,1,1),(0,0,1)\big\}$. ▲

Another interesting case is when an agent is able to remember all the pieces of information she received, and how many times she received them, but fails to memorize their order. In other words, the length of the sequence of pieces of information observed is retained, and so is the frequency of each observed piece of information.[37]

Example 10 (Statistical memory) An agent with statistical memory memorizes a statistical distribution over the set of conceivable pieces of information X, and this justifies our terminology.

For each $n \in \mathbb{N}$, denote as S_n the set of all permutations on $\{1, \ldots, n\}$.[38] Then, agent i exhibits *statistical memory* if $\mathcal{M}_i$ is the correspondence

$$\xi \mapsto \big\{\rho \in X^{\leq N} : \exists\, \pi \in S_{\ell(\xi)},\ \rho = \xi \circ \pi\big\}.$$

The mathematical intuition is as follows. First note that ξ is a map from $\{1, \ldots, n\}$ to X, where we let n denote the length of ξ.[39] Then, each element

[37] Obviously, retaining the number of instances a given piece of information is received implies being able to remember how many pieces of information were received, and this explains why we assume that the length of the sequence is retained.

[38] Our notation comes from the fact that, in abstract algebra and group theory, S_n (endowed with the composition operator) is the symmetric group of $\{1, \ldots, n\}$.

[39] To stress this interpretation, we may denote as $\xi(j)$ instead of x_j the j-th coordinate of ξ (with $j \in \{1, \ldots, n\}$).

$\rho \in \mathcal{M}_i(\xi)$ is obtained by first permuting $\{1,\dots,n\}$ through some $\pi \in S_n$, and then by mapping each permuted element to X through ξ, obtaining $(\xi(\pi(1)),\dots,\xi(\pi(n)))$. Thus, the elements of $\mathcal{M}_i(\xi)$ are the sequences ρ with the same number of occurrences of each element of X as ξ.

For an illustration, consider $\{0,1\}^{\leq 3}$, and $\xi = (1,0,1)$. If agent i exhibits statistical memory, $\mathcal{M}_i(\xi) = \{(1,1,0),(1,0,1),(0,1,1)\}$. ▲

If we allow an agent to remember the pieces of information she observed but not their frequencies we obtain a more imprecise information storage than that of statistical memory.

Example 11 (Range memory) For a generic function $g : X \to Y$, its *range* is the image of its domain. That is, $\operatorname{range} g := g(X)$. Then, we say that agent i satisfies *range memory* if $\mathcal{M}_i$ is the correspondence

$$\xi \mapsto \left\{\rho \in X^{\leq N} : \operatorname{range}\rho = \operatorname{range}\xi\right\}.$$

In words, agent i retains the range of the observed sequence of pieces of information. Hence, as mentioned, the agent is able to memorize which pieces of information she observed, but not their frequencies. Thus, it is not required that the agent correctly retain the length of the observed sequence, which can be imposed as an additional assumption.

For example, consider $\{0,1\}^{\leq 3}$, and $\xi = (1,0,1)$: with range memory, $\mathcal{M}_i(\xi) = \{(0,1),(1,0),(1,1,0),(1,0,1),(0,1,1),(0,0,1),(0,1,0),(1,0,0)\}$. ▲

We may also allow an agent to retain sequences of pieces of information that are somewhat "similar" to what she actually observed.

Example 12 (Fuzzy memory) Suppose that the set X is equipped with some distance d. Then, X^n $(n \in \mathbb{N})$ can be endowed with the metric d_1 defined as follows: for each $\alpha = (a_j)_{j=1}^n, \beta = (b_k)_{k=1}^n \in X^n$, $d_1(\alpha,\beta) := \sum_{j=1}^m d(a_j,b_j)$. With some abuse, we may refer to d_1 as a metric on $X^{\leq T}$, which is however defined only between sequences of the same length.

For each $\varepsilon \in \mathbb{R}_+$, we say that agent i exhibits *ε-fuzzy memory* if $\mathcal{M}_i$ is the correspondence

$$\xi \mapsto \{\rho \in X^{\leq T} : d_1(\xi,\rho) \leq \varepsilon\}.$$

In words, the sequence ξ of pieces of information that is actually observed leads the agent to recall sequences that are "close enough" to ξ. To put it differently, ξ provides the agent with some evidence, but agent i's ability to store information to be "fuzzy": she retains information that is close to (but not necessarily equal to) what she observed.[40]

Note that we could use alternative distances to formulate the notion of fuzzy memory. Specifically, we could consider d_2 and d_∞, which are defined for each $\alpha = (a_j)_{j=1}^n, \beta = (b_k)_{k=1}^n \in X^{\leq T}$ with $\ell(\alpha) = \ell(\beta)$ as $d_2(\alpha, \beta) = (\sum_{j=1}^n (d(a_j, b_j))^2)^{1/2}$ and $d_\infty(\alpha, \beta) = \max_{j \in \{1,\ldots,n\}} d(a_j, b_j)$. Note that perfect memory is in any case obtained by setting $\varepsilon = 0$. ▲

Note that the correspondences proposed so far are partitional. In particular, they are such that the agent never rules out the observed sequence. However, we may also think of more severe failures in the process of information storage.

Example 13 (Information distortion) Agent i *distorts information* if $\mathcal{M}_i$ is not reflexive. That is, if there exist $\xi \in X^{\leq N}$ such that $\xi \notin \mathcal{M}_i(\xi)$. This amounts to assuming that agent i's stored information does not necessarily include the actual experienced stream of pieces of information. ▲

The ability to retain information seems to be influenced by the emotional valence of information, or on an individual's ability to establish a link between the information received and some emotionally-relevant aspect or event. If we allow for set X to be richer, this mechanisms can be captured by memory correspondences.

Example 14 (Emotional memory) Assume that $X = (A^\star \cup M^\star) \times E$, where $E = \{0, 1\}$. For each observed $(x, e) \in X$, we say that x has (positive or negative) emotional valence if $e = 1$. That is, the pieces of information observed by an agent are now enriched by an admittedly rough description of whether they hold some sort of emotional valence for the agent.

We say that agent i exhibits emotional memory if $\mathcal{M}_i$ is the correspondence

$$(x_k, e_k)_{k=1}^n \mapsto \{(y_k, e_k)_{k=1}^n \in X^{\leq T} : \forall\, k \in \{1, \ldots, n\},\ e_k = 1 \Longrightarrow x_k = y_k\}.$$

In words, agent i retains only the observed pieces of information that bear some emotional valence for her or whose observation was associated with a relevant emotional state, as well as the length of the observed sequence.[41]

[40] Fuzzy memory could be though of as describing some features of the process of information retrieval as well. In such case, the observed sequence of information may be seen as a cue that leads an agent to deem possible similar sequences. However, we stick to our preferred interpretation of memory correspondences as possibility correspondences, and from this point of view it is more natural to think of fuzzy memory as a slightly imprecise way of storing information.

[41] The last assumption can easily be relaxed.

Obviously, a more detailed account of emotions may be conceived. For instance, one could posit a richer spectrum of emotional states, ranging from negative to positive ones, and then allow for an individual to remember only the pieces of information that were associated with an emotional state that is similar to the one being currently experienced. With such formalism, for instance, happy individuals would be able to retrieve from their memory those pieces of information they observed when they were happy. Proceeding in this way would allow to shift the emphasis from the storage to the retrieval of information. ▲

We conclude by mentioning a slight modification of the approach presented so far that is achieved by endowing the set of conceivable pieces of information with a different structure. Specifically, we can partition the set of conceivable pieces of information in several categories, representing different kinds of information (e.g., actions taken versus messages received, or even messages pertaining to different topics). Our analysis in the next section will be based on such approach. That is, the sequence of pieces of information available to a given agent will be a sequence of pairs of messages and actions, which obviously represent two different sources of information.

Example 15 (Memory and categories) Suppose that the set X is endowed with a finite partition $\mathscr{C} = \{C_1, C_2, \ldots, C_K\}$, whose elements are called *categories*. Building on the formalism used so far, the set X may be partitioned into two categories, one being that of "actions" and the other one being that of "messages", but we can also imagine finer partitions (e.g., one could envisage categories such as "political news" or "football results" in which messages may be further partitioned).

Then, consider the set $\tilde{X} := \bigtimes_{k \in K} C_k$. An element of such set may be thought of as the information received in some situation, divided into the relevant categories. Obviously, an agent need not be exposed to information from all categories at the same time, so we might as well assume that each category includes a dummy element that specifies that no information from such category is observed at some point. For example, if $\mathscr{C} = \{A^\star, M^\star\}$, it may well be that in some instances a given agent is exposed to some information (i.e., receives a message) without necessarily taking an action.

As before, it is natural to consider the set $\tilde{X}^{\leq N}$ (with $N \in \mathbb{N}$), which represents a sequence of (categorized) information a given agent may receive. The product structure given to set $\tilde{X}$ is then convenient to allow an agent to treat information of different categories in different ways. For an illustration, assume that $\mathscr{C} = \{A^\star, M^\star\}$. If we want agent i to correctly memorize the actions she took but not the messages she received, we can let $\mathcal{M}_i$ be the correspondence $\xi \mapsto \{\rho \in \tilde{X}^{\leq N} : \mathrm{proj}_{(A^\star)^{\leq N}} \rho = \mathrm{proj}_{(A^\star)^{\leq N}} \xi\}$.

Moreover, virtually all the examples mentioned so far can be analyzed within this category-enriched approach. Statistical memory could then be used to formalize the idea that an agent memorizes the frequencies with which she receives information belonging to each category. Or, fuzzy memory could be leveraged to allow for more or less precise storage of different kinds of information. Lastly, note that the emotional message described in Example 14 is formally a special case of this approach, where pieces of information are partitioned based on whether they are emotionally relevant or not. ▲

7.3 Memory in games

The memory correspondences just introduced fit naturally in our framework. If we fix a flow-based game structure $\Gamma_F = \langle I, (A_i, M_i, f_i, \mathcal{A}_i)_{i\in I}\rangle$, we can consider a profile of memory correspondences $(\mathcal{M}_i)_{i\in I}$ such that, for each $i \in I$, $\mathcal{M}_i$ is a serial (i.e., nonempty-valued) possibility correspondence defined over the set $(A_i \cup M_i)^{\leq 2T+1}$.[42] The obtained structure $\langle \Gamma_F, (\mathcal{M}_i)_{i\in I}\rangle$ is a **flow-based game with memory**.

When we derived the standard game structure $\mathrm{Std}(\Gamma_F)$ induced by a flow-based game structure Γ_F, we leveraged the concept of indistinguishable plays to construct information sets (cf. Sections 5.1 and 6.3). To make sense of such derivation, we mentioned the informal interpretive assumption that players are able to memorize the pieces of information they receive. With the theoretical apparatus just defined, we can make justice to such assumption: it can be rigorously stated by appending a profile of memory correspondences satisfying perfect memory to the flow-based game structure of interest.

To see this in detail, consider a flow-based game structure with memory $\langle \Gamma_F, (\mathcal{M}_i)_{i\in I}\rangle$, as well as the set of plays P derived from Γ_F. Recall that two plays $p, q \in P_i$ are **indistinguishable** for player $i \in I$, written $p \sim_i q$, if and only if $F_i(p) = F_i(q)$ (cf. Sections 5.1 and 6.3). Two plays are indistinguishable for a player if they induce the same realized stream of information for such player. Importantly, such stream is entirely determined by the rules of the game. Now, for any two plays $p, q \in P_i$, we say that q is **mistakable for** p by player $i \in I$, written $p \rightsquigarrow_i q$, if $F_i(q) \in \mathcal{M}_i(F_i(p))$. In words, play q can be mistaken for (or confused with) play p if the stream of

[42]Specifically, the sequences of pieces of information a player may get to observe according to the rules of the game are such that (i) their length is odd, and (ii) they map odd numbers to M_i and even numbers to A_i.

information it induces (i.e., $F_i(q)$) is consistent with what player i recalls of the flow induced by p (i.e., it belongs to $\mathcal{M}_i(F_i(p))$).

It is worth noting that indistinguishability is an *objective* notion: two plays are indistinguishable if the rules of the game themselves do not allow a player to distinguish between them. On the other hand, mistakability is *subjective*, as it arises from a player's cognitive features.

At this point, we can mimic the construction of Sections 5.1 and 6.3 to derive what we call a "generalized standard game structure" $\langle I, V, \mathcal{I}, (A_i, \mathscr{M}_i)_{i\in I}\rangle$ from a flow-based game structure with memory $\langle \Gamma_F, (\mathcal{M}_i)_{i\in I}\rangle$. Specifically, the derivation of V and $\mathcal{I}$ is as in Section 6.3. Here, however, we retrieve sets of mistakable plays of each player $i \in I$ through the relation $\rightsquigarrow_i$, as we want to explicitly account for subjective cognitive features. Note that the resulting collection of sets *need not be* a partition.[43] Hence, for each $i \in I$ and for each play $v \in V_i$ at which i is alert, let $M_i(v) := \{w \in V_i : v \rightsquigarrow_i w\}$ be the set of plays v may be mistaken for by i. Then, define

$$\mathscr{M}_i := \{M_i(v) \subseteq V_i : v \in V_i\}.$$

Note that $\mathscr{M}_i$ ($i \in I$) is a partition whenever $\mathcal{M}_i$ is partitional.[44] Moreover, condition (KfA-G) of Definition 8 need not hold in general.

We denote as $\mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I})$ the structure $\langle I, V, \mathcal{I}, (A_i, \mathscr{M}_i)_{i\in I}\rangle$ obtained as just described from Γ_F and $(\mathcal{M}_i)_{i\in I}$, and we call it the **generalized standard game structure** induced by the flow-based game structure with memory $\langle \Gamma_F, (\mathcal{M}_i)_{i\in I}\rangle$. We use the word "generalized" because, as mentioned, the collections $(\mathscr{M}_i)_{i\in I}$ need not be partitional, and condition (KfA-G) of Definition 8 may fail. If these two conditions are both satisfied, however, $\mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I})$ is a standard game structure.

As a matter of terminology, for a profile of memory correspondences $(\mathcal{M}_i)_{i\in I}$, we say that a property holds for the profile $(\mathcal{M}_i)_{i\in I}$ if it holds for each $\mathcal{M}_i$ with $i \in I$. Now note that, if $\mathcal{M}_i$ satisfies perfect memory, two plays after which i is alert can be mistaken for each other if and only if they are indistinguishable.[45] The following result gives a formal justification of the interpretive assumption we mentioned to make sense of the constructions of Sections 5.1 and 6.3, as well as of Propositions 1 and 3.

[43]In contrast, in Section 6.3, we used the (objective) indistinguishability relation $\sim_i$ ($i \in I$) to retrieve the information sets of player i. Since $\sim_i$ is an equivalence relation, the resulting collection of sets was ensured to be a partition of the set of plays where player i is alert.

[44]In such case, $\rightsquigarrow_i$ is an equivalence relation and $\mathscr{M}_i$ is simply the quotient space $V_i/\rightsquigarrow_i$.

[45]Formally, whenever $\mathcal{M}_i$ satisfies perfect memory, $P_i/\sim_i = V_i/\rightsquigarrow_i$.

Proposition 5 *Fix a standard game structure* Γ_S. *The perfect recall property* (PR) *holds in* Γ_S *if and only if there exist a flow-based game structure* Γ_F *and a profile of memory correspondences* $(\mathcal{M}_i)_{i\in I}$ *satisfying perfect memory such that* $\Gamma_S = \mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I})$.

Unsurprisingly, Proposition 5 ensures that a standard game structure satisfying perfect recall can be retrieved from a flow-based game structure with memory where the memory correspondences satisfy perfect memory. Conversely, if a standard game structure satisfies perfect recall, then it must be retrievable from some flow-based game structure with memory where perfect memory holds. This is yet another way to clarify that information sets *do* in general entail assumptions about players' cognitive abilities. However, we can say more: *any* standard game structure can be retrieved from a suitable flow-based game structure with memory where the memory correspondences are partitional.

Proposition 6 *Fix a standard game structure* Γ_S. *There exist a flow-based game structure* Γ_F *and a profile of partitional memory correspondences* $(\mathcal{M}_i)_{i\in I}$ *such that* $\Gamma_S = \mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I})$.

Note that Proposition 6 ensures that any situation that can be described by a standard formalism can be framed within our memory-enriched approach. On the other hand, there are situations that are easily described with our language while at the same time being inexpressible in a standard setting. Indeed, consider a flow-based game structure with memory $\langle \Gamma_F, (\mathcal{M})_{i\in I}\rangle$, and suppose that player $i \in I$ distorts information (cf. Example 13). The resulting mistakability relation for i is *not* an equivalence relation, as it fails reflexivity. Hence, the $\mathscr{M}_i$ is not a partition, and this makes it impossible to retrieve a standard game structure from $\langle \Gamma_F, (\mathcal{M})_{i\in I}\rangle$. Intuitively, information distortion implies that, when some play realizes, player i deems such play impossible: this is obviously inexpressible in a standard setting, where each player's information structure is a partition of the set of plays where such player is alert.

It is worth stressing that Proposition 6 applies to any game structure, and this includes of course those where perfect recall fails, as well as those encoding "problematic" cognitive failures such as absentmindedness.

Example 16 (Absentminded driver) The following game structure is taken from Piccione and Rubinstein (1997). A driver is heading back home and he needs to get off the highway at the second exit. However, when he reaches an exit, he cannot understand if it is the first or the second one –

that is, he does not remember if such exit is the first one he crosses or if he already passed past one. The graphical representation is as follows.

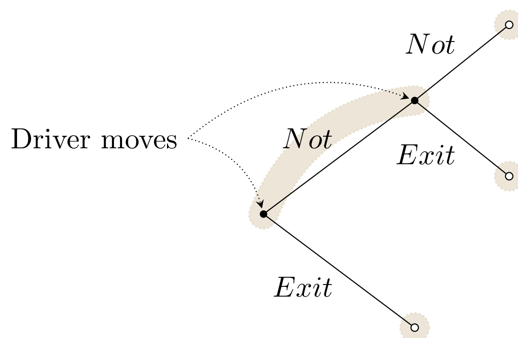

Figure 5 Driver's information sets.

To retrieve a flow-based game structure with memory inducing such situation, focus on the following flow of information.

Play	**Message**	**Available actions**
$\varnothing$	$m^* =$ *You are at a crossing.*	$\{Exit, Not\}$
$(Exit)$	$m_{\mathrm{w}} =$ *Wrong exit!*	$\emptyset$
(Not)	$m^* =$ *You are at a crossing.*	$\{Exit, Not\}$
$(Not, Exit)$	$m_{\mathrm{h}} =$ *You arrived home!*	$\emptyset$
(Not, Not)	$m_{\mathrm{m}} =$ *You missed your exit!*	$\emptyset$

Table 3 A flow of information for Driver.

The feasible extended histories are straightforwardly derived. Moreover it is easy to verify that distinct plays are always distinguishable. For instance, consider $\varnothing$ and (Not): their induced streams of information are $F_D(\varnothing) = (m^*)$ and $F_D(Not) = (m^*, Not, m^*)$, which are obviously different. Hence, confusion between such two plays *cannot* arise as a byproduct of the rules of the game. Rather, we have to introduce a memory correspondence $\mathcal{M}_D$ such that Driver only retains the last message induced by a play, and nothing more. Specifically, this implies that $\mathcal{M}_D((m^*)) = \mathcal{M}_D((m^*, Not, m^*)) = \{(m^*), (m^*, Not, m^*)\}$. Therefore, despite being sometimes regarded as a pathological feature that can arise from a flawed specification of the information structure of a standard game structure,[46] absentmindedness can be modeled smoothly in our memory-enriched framework. ▲

Propositions 5 and 6 are existence results. This means that one may find many flow-based game structures with memory inducing the same standard game structure. In particular, it is interesting to note that a standard game structure with perfect recall may be induced by a flow-based game structure with memory where the memory correspondences *do not* satisfy perfect memory. The following example illustrates.

Example 17 (A quartet game, continued) Consider the game structure introduced in Example 4, and focus on Dave's standard information structure as depicted by the light blue information sets in Figure 2. To retrieve such information structure, assume that Dave is alert (at the root, and) at the plays after which he is active – namely, (Ann, ℓ), (Ann, r), (Bob, ℓ), and (Bob, r).[47] Then, further assume that he perfectly observes the actions taken by Ann, Bob, and Chloe: we can imagine that the messages he may observe before acting are precisely (Ann, ℓ), (Ann, r), (Bob, ℓ), and (Bob, r). Then, a memory correspondence $\mathcal{M}_D$ that partitions the set $\{(Ann, \ell), (Ann, r), (Bob, \ell), (Bob, r)\}$ into the two equivalence classes $\{(Ann, \ell), (Bob, \ell)\}$ and $\{(Ann, r), (Bob, r)\}$ would result in the light blue information structure of Figure 2.

On the other hand, perfect recall holds in such information structure, so we know from Proposition 5 that such situation may be induced by a suitable flow of information, under the assumption that Dave exhibits perfect memory. In particular, the desired flow of information will inform Dave on the action taken immediately before his turn, but not of who chose it.

Therefore, the situation depicted in such graph may arise in (at least) the following two cases: (i) the rules of the game prevent Dave from knowing who acted before him, but allow him to know which action was chose, and (ii) the rules of the game perfectly inform Dave of the game unfolding, but Dave's cognitive features prevent him from correctly memorizing who acted before him. ▲

The foregoing example and discussion provides another illustration of how the formalism of information sets may fail to comply with the sep-

[46] For instance, absentmindedness is ruled out by the defining features of the modeling framework introduced by Kuhn (1953). See also the discussion in Alós-Ferrer and Ritzberger (2016), pp 75-78.

[47] Recall that in such game tree Chloe first decides whether to give the move to Ann or Bob, and later on the chosen player decides between ℓ and r.

aration principle: while it describes how informed players are during the game unfolding, it does not fully specify where their "knowledge" comes from. The example clarifies that, even when the information structure of a standard game structure is well-behaved enough to satisfy perfect recall, it may arise from different combinations of objectively-determined rules of interaction and subjective cognitive features.

Given the foregoing results, and Propositions 5 and 6, we can make our earlier discussion about the objective and subjective features of the perfect recall property more precise: *under the assumption of perfect memory* (a feature of the agents who happen to play the game), information structures with perfect recall express players' knowledge of the game unfolding as determined by the rules of interaction. If we drop such assumption, perfect recall is no longer justifiable or interpretable in this way, and Example 17 is a case in point.

8 Conclusion

Related literature The present paper relates to several strands of the literature. First of all, it is closely tied to the theory of the detailed representation of sequential games (or, extensive-form games, as they are commonly called).[48] The first definition of "extensive-form games" is due to the seminal work of von Neumann and Morgenstern (1944), who start with a set of "outcomes" that are progressively refined by players' choices. This yields as a derived object a graph-theoretic representation with trees satisfying a multistage structure, i.e., all nodes in the same information set have the same number of predecessors. Information is assumed to have a partitional structure and perfect recall is not a maintained assumption. To remove the built-in multistage assumption of von Neumann and Morgenstern (1944), Kuhn (1953) posited the tree representation as primitive and defined the perfect recall property of information partitions. Alós-Ferrer and Ritzberger (2008, 2013) generalized the representations of von Neumann and Morgenstern (1944) and Kuhn (1953). Like the former, they start with a set of outcomes and let choices select subsets of outcomes, like the latter they do not assume a multistage structure; furthermore, they allow for all kinds of infinite games studied in applications and not impose a partitional information structure. The book by Alós-Ferrer and Ritzberger (2016) provides

[48] We are critical of the "normal/extensive form game" terminology: as von Neumann and Morgenstern (1944) make clear, the normal and extensive form are different kinds of *representations* of games, not different kinds of games.

a broad overview of the field. Finally, seminal work of Harris (1985) introduced the sequence representation later used by Osborne and Rubinstein (1994, Definition 203.3) in their texbook. Like them, we use the sequence representation, but—as we explained at length—, we represent information in a crucially different way.

As for perfect recall, several definitions of the same concept have been proposed in addition to the recent one by Osborne and Rubinstein (1994, Definition 203.3) used here. The first one is due to Kuhn (1953, Definition 17), and it leverages the derived concept of strategy, rather than primitive elements of the analysis. Other notions are proposed by Selten (1975) and Perea (2001, Definition 2.1.2). Alós-Ferrer and Ritzberger (2016, Proposition 6.6) and Alós-Ferrer and Ritzberger (2017, Theorem 1 and Corollary 1) prove the equivalence of all the aforementioned notions. Remarkably, the equivalence continues to hold in games with infinite horizon, and in games where agents can choose their actions from an infinite set.

Other works focused on the interpretation and on the characterizations of perfect recall (Alós-Ferrer & Ritzberger, 2017; Bonanno, 2003, 2004; Okada, 1987; Ritzberger, 1999). The most important insight that comes from this branch of the literature is that perfect recall indeed captures a situation where (*i*) players never forget what they did, (*ii*) players never forget what they knew, and (*iii*), past, present, and future have an unambiguous meaning (Ritzberger, 1999). A similar decomposition of perfect recall in "memory of past knowledge" and "memory of past actions" is obtained by Bonanno (2003, 2004) following a syntactic approach that relies on tools from temporal logic. Our framework provides another perspective to look at the same issues, and our results are complementary to the aforementioned ones.

A burgeoning body of literature analyzed the role of memory in decision problems (Bordalo, Coffman, Gennaioli, Schwerter, & Shleifer, 2020; Bordalo, Gennaioli, & Shleifer, 2019, 2020). Such works underline the role of environmental cues in facilitating the retrieval of similar past experiences from memory, and they show how this process influences an agent's assessment of given information. Our general analysis of memory is somewhat complementary: while this literature focused on the *retrieval* of stored information, we aimed at modeling the process of information *storage*. Given the generality of the approach proposed in the present paper, we believe that blending the two perspectives may be both feasible and insightful.

As already hinted, our approach allows to describe the information that may accrue to alert but inactive players, which is instead usually neglected. This relates our analysis to two strands of the literature. On the one hand, the information an inactive player receives throughout the play may well be relevant for psychological reasons (cf. Battigalli & Dufwenberg, forthcoming for a survey of the literature on psychological games). On the other hand, the end-game information available to player is key when self-confirming equilibrium is studied. In the setting presented so far, such information is the cumulated information available to a given player after some terminal play (i.e., the personal history induced by such terminal play). The literature on self-confirming equilibrium indeed usually posits an end-game feedback function about the play, but, when sequential games are studied, the information available to players during the game is described by means of information sets (cf. Battigalli, Catonini, Lanzani, & Marinacci, 2019). A flow-based description of information may help harmonize such hybrid representation.

Lastly, we mentioned that a flow-based approach has already been used to describe the information accrual to players throughout a game. In particular, in the vast majority of models of repeated games information is modeled as a flow. Specifically, after each round of interaction players obtain a novel piece of information, which may be somewhat revealing of the actions chosen by co-players. For instance, in the oligopoly model of Green and Porter (1984) firms observe the price realization at each period, which is an noisy signal of the competitors' production choices. A sequence of such signals and of chosen actions is the information available to a given firm at a given point in time: the similarity with our approach is obvious. Outside of the literature on repeated games, the same approach was used in Myerson (1986) and Myerson and Reny (2020).

Concluding remarks This paper proposed a framework to explicitly describe players' information in sequential games as provided by the rules of interaction. In doing so, we focused on *flows* of information, as opposed to the standard information-set-based representation which treats information as a *stock*. While this approach is admittedly not new, we tried to offer a systematic exposition, and we enriched it with a formal description of players' ability to retain information. We argued that flows of information provide an explicit and complete description of the information that *objectively* accrues to players in a sequential way as the game unfolds, while the memory correspondences we introduced give a formal definition of players' *subjective* ability to retain the observed information. Decoupling objective and subjective informational aspects allows us to comply with the separation principle, it makes our language more expressive, and it is a key step to introduce cognitive limitations in game theoretic analyses in a formal way.

In this regard, we believe that one of the most promising avenues for future research would consist in trying to enrich our understanding of and our ability to formalize the cognitive aspects linked to memory. This may hopefully help to combine our model of information storage with the already mentioned works on information retrieval given some environmental cues, or to shed light on the emotional aspects influencing mnemonic abilities.

Moreover, it would be interesting to embed such memory-related elements of bounded rationality in interactive situations. In particular, it could be possible to model situations where (not necessarily rational) players wonder about others' cognitive abilities when reasoning strategically. Expressing this kind of assumptions in a formal way would require an expressive language – in particular, one should work with a rich space of states of the world, where irrationality and cognitive failures are allowed to persist at some states.[49] We believe this to be crucial for a better understanding of the implications of bounded rationality and cognitive limitations in strategic interactions.

Lastly, it is worth noting that several of our assumptions may be relaxed in a straightforward way. In particular, we can seamlessly allow for infinite sets of actions and messages, as well as for an infinite length of the game of interest: this would imply virtually no changes as far as definitions are concerned, while some additional requirements may be needed to generalize our results.[50] We can also allow for stochastic elements in a relatively simple way: it is enough to allow for some chance moves by including chance in the set of players. Given that we are interested in the representation of a game structure with possibly some chance moves, we are not even required to specify which are the probabilities of such moves. Note that this device could allow us to make players' mnemonic abilities depend on stochastic elements.

A Proofs

A.1 Proof of Lemma 1 (p. 18)

Fix a standard multistage game structure $\Gamma_{SM} = \langle I, V, (A_i, \mathscr{Q}_i)_{i \in I} \rangle$ satisfying perfect recall, and denote its completion as $\Gamma_{SI} = \langle I, H, (A_i, \mathscr{R}_i)_{i \in I} \rangle$. We begin by noting that, as per condition (KS-SI), two plays $v, w \in V$ may

[49] This formalism is employed by Battigalli, Corrao, and Sanna (2020), even though attention is restricted to rational players there.

[50] Specifically, one should impose appropriate measurability conditions to the feedback functions.

belong to the same information set of a player only if they have the same length. Hence, we proceed by induction on the length of v and w to show (PR-SI). The statement trivially holds for $\ell(v) = \ell(w) = 0$. Assume that it holds for $\ell(v) = \ell(w) \le k \in \mathbb{N}_0$. Then, suppose that $\ell(v) = \ell(w) = k+1$ and $R_i(v) = R_i(w)$. By definition of experience, we have

$$X_i(v) = (X_i(\operatorname{pre} v), a_i(\operatorname{pre} v, v), R_i(v)), \quad X_i(w) = (X_i(\operatorname{pre} w), a_i(\operatorname{pre} w, w), R_i(w)).$$

Now, note the following. First, conditions (NF) and (KS-SI) ensure that $X_i(\operatorname{pre} v) = X_i(\operatorname{pre} w)$. To see why this holds, consider play $x \in V$ and its immediate predecessor $\operatorname{pre} x$. Clearly, $\operatorname{pre} x \prec x$, and (NF) implies

$$\forall\, x' \in R_i(x), \exists\, y \in R_i(\operatorname{pre} x),\ y \prec x'.$$

Then, given that both $\operatorname{pre} x$ and y belong to the same information set $R_i(\operatorname{pre} x)$, condition (KS-SI) implies that they have the same length – that is, $\ell(\operatorname{pre} x) = \ell(y)$. In particular, such length will obviously be equal to $\ell(x) - 1 = \ell(x') - 1$. Finally, note that there is only one predecessor of x' of length $\ell(x') - 1$, so it must be the case that $y = \operatorname{pre} x'$. All in all, conditions (NF) and (KS-SI) allow to conclude that, if two plays belong to the same information set, then so do their immediate predecessors. In light of this, we can say that $R_i(\operatorname{pre} v) = R_i(\operatorname{pre} w)$, and given that $\ell(\operatorname{pre} x) = \ell(\operatorname{pre} w) \le k$, $X_i(\operatorname{pre} v) = X_i(\operatorname{pre} w)$ by the inductive hypothesis.

Second, condition (ROA) straightforwardly implies that, $a_i(\operatorname{pre} v, v) = a_i(\operatorname{pre} w, w)$ since $R_i(v) = R_i(w)$.

Third, $R_i(v) = R_i(w)$ by assumption.

Wrapping up, the foregoing observations show that $X_i(v) = X_i(w)$, and this concludes the induction. ■

A.2 Proof of Proposition 1 (p. 20)

Proposition 1 is a special case of Proposition 3, so we refer the reader to Section A.4 for the general proof.

A.3 Proof of Proposition 2 (p. 22)

Proposition 2 is a special case of Proposition 4, so we refer the reader to Section A.5 for the general proof.

A.4 Proof of Proposition 3 (p. 33)

Fix a player $i \in I$, and $p, q \in P = V$ such that $U_i(p) = U_i(q)$. We may as well assume that $p \neq \varnothing$ and $q \neq \varnothing$, as (PR) trivially holds for the empty play (given that $U_i(\varnothing) = \{\varnothing\}$). Now note that $U_i(p) = U_i(q)$ means that $F_i(p) = F_i(q)$, which in turn implies that $\ell(F_i(p)) = \ell(F_i(q))$. We can then proceed by induction on such length.

Basis step. Assume that $\ell(F_i(p)) = \ell(F_i(q)) = 1$. This implies that $\mathrm{last}_i\, p = \mathrm{last}_i\, q = \varnothing$. Hence, $F_i(p) = (\mathrm{proj}_{M_i} f(\varnothing), a_i(\varnothing, p))$ and $F_i(q) = (\mathrm{proj}_{M_i} f(\varnothing), a_i(\varnothing, q))$. Since $F_i(p) = F_i(q)$, we conclude that $a_i(\varnothing, p) = a_i(\varnothing, q)$. At this point, note that

$$
\begin{aligned}
X_i(p) &= (X_i(\mathrm{last}_i\, p), a_i(\mathrm{last}_i\, p, p), U_i(p)) = (\{\varnothing\}, a_i(\varnothing, p), U_i(p)),\\
X_i(q) &= (X_i(\mathrm{last}_i\, q), a_i(\mathrm{last}_i\, q, q), U_i(q)) = (\{\varnothing\}, a_i(\varnothing, q), U_i(q)).
\end{aligned}
$$

Since $a_i(\varnothing, p) = a_i(\varnothing, q)$ as already mentioned, and $U_i(p) = U_i(q)$ by assumption, we obtain $X_i(p) = X_i(q)$.

Inductive step. Assume that the claim holds for $\ell(F_i(p)) = \ell(F_i(q)) = k \in \{1, \dots, t\}$, and focus on $p, q \in P$ such that $\ell(F_i(p)) = \ell(F_i(q)) = t + 1 \in \mathbb{N}$ and $Q_i(p) = Q_i(q)$. Note that $U_i(p) = U_i(q)$ implies that $U_i(r) = U_i(s)$ for each $r \preceq p$ and $s \preceq q$ where player i is alert. Given that $\mathrm{last}_i\, p \preceq p$ and $\mathrm{last}_i\, q \preceq q$, we have $U_i(\mathrm{last}_i\, p) = U_i(\mathrm{last}_i\, q)$. Since $\ell(\mathrm{last}_i\, p) = \ell(\mathrm{last}_i\, q) \leq t$, the inductive hypothesis implies that $X_i(\mathrm{last}_i\, p) = X_i(\mathrm{last}_i\, q)$. Now note that

$$
\begin{aligned}
X_i(p) &= (X_i(\mathrm{last}_i\, p), a_i(\mathrm{last}_i\, p, p), U_i(p)),\\
X_i(q) &= (X_i(\mathrm{last}_i\, q), a_i(\mathrm{last}_i\, q, q), U_i(q)).
\end{aligned}
$$

We already showed that $X_i(\mathrm{last}_i\, p) = X_i(\mathrm{last}_i\, q)$. Moreover, $U_i(p) = U_i(q)$ by assumption. Lastly, $a_i(\mathrm{last}_i\, p, p) = a_i(\mathrm{last}_i\, q, q)$ because $F_i(p) = F_i(q)$. Therefore, $X_i(p) = X_i(q)$, and this concludes the proof. ■

A.5 Proof of Proposition 4 (p. 33)

"Only if" direction. Consider $\Gamma_S = \langle I, V, \mathcal{I}, (A_i, \mathscr{U}_i)_{i \in I} \rangle$ satisfying perfect recall. We begin by noting that Definition 9 and the definition of experience functions allow to conclude that the following holds in Γ_S:

$$
\forall\, i \in I, \forall\, v, w \in V_i, \quad U_i(v) = U_i(w) \Longleftrightarrow X_i(v) = X_i(w). \qquad (\mathrm{U} {\leftrightarrow} \mathrm{X})
$$

Now, we want to construct a flow-based game structure $\Gamma_F = \langle I, \tilde{f}, (A_i, M_i, \mathcal{A}_i)_{i \in I} \rangle$ such that $\mathrm{Std}(\Gamma_F) = \Gamma_S$. We begin with the construction.

First of all, for each $i \in I$, let $M_i := \mathscr{U}_i$.

Then, define $\tilde{f}_i : A^{\leq T} \to M_i$ to be, for each $a^t \in A^{\leq T}$,

$$\tilde{f}_i(a^t) := \begin{cases} U_i(a^t) & \text{if } a^t \in V_i; \\ U_i & \text{otherwise;} \end{cases}$$

where U_i is a generic element of $\mathscr{U}_i$. Now let $\tilde{f} : A^{\leq T} \to M$ be, for each $a^t \in A^{\leq T}$,

$$\tilde{f}(a^t) := \begin{cases} (\tilde{f}_i(a^t))_{i \in \mathcal{I}(a^t)} & \text{if } a^t \in V; \\ (f_i(a^t))_{i \in I} & \text{otherwise.} \end{cases}$$

Moreover, for each $i \in I$ define $\mathcal{A}_i : M_i \rightrightarrows A_i$ to be such that, for each $m_i \in M_i$,

$$\mathcal{A}_i(m_i) := \operatorname{proj}_{A_i} A(w),$$

where w is a generic element of $\{v \in V_i : U_i(v) = m_i\}$. Note that player i's feasible actions ($i \in I$) are the same for each play in such set thanks to property (KfA), so that $\mathcal{A}_i$ is always well-defined.

Consider now $\Gamma_F := \langle I, \tilde{f}, (A_i, M_i, \mathcal{A}_i)_{i \in I} \rangle$, where M_i, $\tilde{f}$ and $(\mathcal{A}_i)_{i \in I}$ are defined in the way just described. It is now time to check that $\mathrm{Std}(\Gamma_F) = \Gamma_S$. To ease the comparison, write $\mathrm{Std}(\Gamma_F) = \langle I, \tilde{V}, \tilde{\mathcal{I}}, (A_i, \tilde{\mathscr{U}}_i)_{i \in I} \rangle$, and recall that $\Gamma_S = \langle I, V, \mathcal{I}, (A_i, \mathscr{U}_i)_{i \in I} \rangle$. We want to prove that (i) $V = \tilde{V}$, (ii), $\mathcal{I} = \tilde{\mathcal{I}}$, and (iii) $(\mathscr{U}_i)_{i \in I} = (\tilde{\mathscr{U}}_i)_{i \in I}$.

From Γ_F we can derive P, H, and f in a standard way (cf. Section 6.1). Obviously, $P = \tilde{V}$, but is is straightforward to check that also $P = V$, and this yields (i). Moreover, the alert-player correspondence $\tilde{\mathcal{I}} : V \rightrightarrows I$ is $v \mapsto \mathcal{D}(\tilde{f}(v))$ by definition (cf. Section 6.3). But, by definition of $\tilde{f}$, $\mathcal{D}(\tilde{f}(v)) = \mathcal{I}(v)$ for each $v \in V$, and therefore (ii) is met as well.

To check condition (iii), we proceed in two steps. The way in which we defined the feedback function $\tilde{f}$ implies the following:

$$\forall\, i \in I, \forall\, v, w \in V_i, \quad X_i(v) = X_i(w) \Longleftrightarrow F_i(v) = F_i(w). \tag{X$\leftrightarrow$F}$$

To see why this holds, fix $i \in I$ and $v, w \in V_i$. Then:

- Assume $X_i(v) = X_i(w)$. This implies $\ell(X_i(v)) = \ell(X_i(w))$, so we can proceed by induction on such length.
  Basis step. Assume $\ell(X_i(v)) = \ell(X_i(w)) = 1$. This can only happen if $v = w = \varnothing$, so that $X_i(v) = X_i(w) = (\{\varnothing\})$. Then, $F_i(v) = F_i(w)$ trivially holds as $v = w$.
  Inductive step. Now assume that the claim holds for $\ell(X_i(v)) =$

$\ell(X_i(w)) \in \{1, \dots, n\}$ for some $n \in \mathbb{N}$, and suppose $\ell(X_i(v)) = \ell(X_i(w)) = n + 1$. By perfect recall, it can be checked that $X_i(t) = X_i(u)$ for each $t \preceq v$ and $u \preceq w$ where player i is alert. Hence, $X_i(\operatorname{last}_i v) = X_i(\operatorname{last}_i w)$, which implies $F_i(\operatorname{last}_i v) = F_i(\operatorname{last}_i w)$ thanks to the inductive hypothesis (as $\ell(X_i(\operatorname{last}_i v)) = \ell(X_i(\operatorname{last}_i w)) \leq n$). Therefore,

$$(X_i(\operatorname{last}_i v), a_i(\operatorname{last}_i v, v), U_i(v)) = (X_i(\operatorname{last}_i w), a_i(\operatorname{last}_i w, w), U_i(w)),$$

which implies $a_i(\operatorname{last}_i v, v) = a_i(\operatorname{last}_i w, w)$ and $U_i(v) = U_i(w)$. By construction of $\tilde{f}$, $U_i(x) = \tilde{f}_i(x) = f_i(x)$ for $x \in \{v, w\}$, and this allows to say that $f_i(v) = f_i(w)$. Wrapping up, we write

$$\begin{aligned} F_i(v) &= (F_i(\operatorname{last}_i v), a_i(\operatorname{last}_i v, v), f_i(v)), \\ F_i(w) &= (F_i(\operatorname{last}_i w), a_i(\operatorname{last}_i w, w), f_i(w)), \end{aligned}$$

and we conclude that $F_i(v) = F_i(w)$ in light of the foregoing observations.

- Assume $F_i(v) = F_i(w)$. Again, we proceed by induction on $\ell(F_i(v)) = \ell(F_i(w)) \in \mathbb{N}_0$.
  Basis step. The result trivially holds for $\ell(F_i(v)) = \ell(F_i(w)) = 0$.
  Inductive step. By arguments analogous to the proof above.

At this point, (U↔X) and (X↔F) yield

$$\forall\, i \in I, \forall\, v, w \in V_i, \quad U_i(v) = U_i(w) \iff F_i(v) = F_i(w).$$

Given that, for each $i \in I$ and $v, w \in V_i$, $F_i(v) = F_i(w)$ is equivalent to $v \sim_i w$, it is also equivalent to $\tilde{U}_i(v) = \tilde{U}_i(w)$, and this is easily verified by inspecting the way in which we derived the information sets of the standard game structure induced by a flow-based one. This establishes point (*iii*).

In light of the foregoing observations, we can say that $\operatorname{Std}(\Gamma_F) = \Gamma_S$, and this concludes the proof of the "only if" direction of the statement.

"If" direction. Fix a standard game structure Γ_S and assume that $\Gamma_S = \operatorname{Std}(\Gamma_F)$ for some flow-based game structure Γ_S. By Proposition 3, $\operatorname{Std}(\Gamma_F)$ satisfies prefect recall, hence so does Γ_S. ■

A.6 Proof of Proposition 5 (p. 42)

We first report the following fact, which is rather straightforward.

Lemma A1 *Fix a flow-based game structure* Γ_F *and a profile of memory correspondences* $(\mathcal{M}_i)_{i\in I}$. *If* $(\mathcal{M}_i)_{i\in I}$ *satisfies perfect memory,* $\mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I}) = \mathrm{Std}(\Gamma_F)$.

Proof of Lemma A1. Note that perfect memory implies that $\mathcal{M}_i$ is partitional for each $i \in I$. Consistently with the notation used in earlier sections, let $\mathscr{M}_i$ $(i \in I)$ denote the partition of mistakable plays for player i in the generalized standard game tree $\mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I})$, as derived in Section 7.3. Also, let $\mathscr{U}_i$ ($i \in I$ denote player i's information structure in the standard game tree $\mathrm{Std}(\Gamma_F)$.

To prove the result, we show that, for each $i \in I$, $\mathscr{M}_i = \mathscr{U}_i$. To this end, fix a generic $i \in I$. Then, it is enough to note that perfect memory implies that $V_i/\sim_i = V_i/\rightsquigarrow_i$, where V_i is the set of plays where i is alert. Given that $V_i/\sim_i = \mathscr{U}_i$ and $V_i/\rightsquigarrow_i = \mathscr{M}_i$, the desired result follows. ■

We now proceed with the proof of Proposition 5.

"Only if" direction. Consider a standard game structure $\Gamma_S = \langle I, V, \mathcal{I}, (A_i, \mathscr{U}_i)_{i\in I}\rangle$ satisfying (PR). By Proposition 3, there exists a flow-based game structure Γ_F such that $\Gamma_S = \mathrm{Std}(\Gamma_F)$. By Lemma A1, $\mathrm{Std}(\Gamma_F) = \mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I})$, where $(\mathcal{M}_i)_{i\in I}$ satisfies perfect memory. Hence, $\Gamma_S = \mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I})$, and this proves this direction of the statement.

"If" direction. Suppose that $\Gamma_S = \mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I})$ for some flow-based game structure Γ_F and profile of memory correspondences $(\mathcal{M}_i)_{i\in I}$ satisfying perfect memory. By Lemma A1, $\mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I}) = \mathrm{Std}(\Gamma_F)$. By Proposition 3, $\mathrm{Std}(\Gamma_F)$ satisfies (PR), and this concludes the proof. ■

A.7 Proof of Proposition 6 (p. 42)

Consider a standard game structure $\Gamma_S = \langle I, V, \mathcal{I}, (A_i, \mathscr{U}_i)_{i\in I}\rangle$. We want to construct a flow-based game structure $\Gamma_F = \langle I, \tilde{f}, (A_i, M_i, \mathcal{A}_i)_{i\in I}\rangle$ inducing V as the set of feasible plays, where each play induces a different stream of information. Then, it is enough to consider a partitional profile of memory correspondences $(\mathcal{M}_i)_{i\in I}$ such that, for each $i \in I$ and $v \in V_i$, $U_i(v) = \mathcal{M}_i(F_i(v)) = [\,v\,]_{\rightsquigarrow_i}$: this implies that $\Gamma_S = \mathrm{GStd}(\Gamma_F, (\mathcal{M}_i)_{i\in I})$, yielding the desired result. In the following, we only show the existence of a suitable flow-based game structure.

As a matter of notation, denote as $V^t \subseteq A^t$ $(t \in \{0, \dots, T\})$ the set of t-long plays in V: formally $V^t := \{v \in V : \ell(v) = t\}$. Similarly, let $V_i^t \subseteq A^t$ $(i \in I,\ t \in \{0, \dots, T\})$ the set of t-long plays in V after which player i is alert: formally $V_i^t := \{v \in V_i : \ell(v) = t\}$.

Step 1: construction of the flow-based game structure. To construct the desired flow-based game structure, we proceed as follows. First, for each $i \in I$, let $M_i := A^{\leq T}$. Second, let $\tilde{f} : A^{\leq T} \to A^{\leq T}$ be the map defined for each $a^t \in A^{\leq T}$ as

$$\tilde{f}(a^t) := \begin{cases} (a^t)_{i \in \mathcal{I}(a^t)} & \text{if } a^t \in V; \\ (a^t)_{i \in I} & \text{otherwise.} \end{cases}$$

Third, for each $i \in I$, let $\mathcal{A}_i$ be defined for each $a^t \in A^{\leq T}$ as

$$\mathcal{A}_i(a^t) := \begin{cases} \operatorname{proj}_{A_i} A(a^t) & \text{if } a^t \in V_i; \\ \emptyset & \text{otherwise.} \end{cases}$$

Step 2: verify that $P = V$. Let P be the set of feasible plays induced by the flow-based game structure $\Gamma_F = \langle I, \tilde{f}, (A_i, M_i, \mathcal{A}_i)_{i \in I} \rangle$. We proceed by induction on the length of such plays. Let $P^t := \{p \in P : \ell(p) = t\}$ $(t \in \{0, \ldots, T\})$ be the set of feasible plays with length t.

Then, $P^0 = \{\varnothing\} = V^0$ holds trivially. Assume by way of induction that $P^t = V^t$ for $t \in \{0, \ldots, T-1\}$: we want to show that $P^{t+1} = V^{t+1}$.

Consider $p = (\operatorname{pre} p, a) \in P^{t+1}$. We have that *(i)* $\mathcal{D}(a) = \mathcal{D}(\tilde{f}(\operatorname{pre} p)) = \mathcal{I}(\operatorname{pre} p)$, and *(ii)* $a \in \bigtimes_{i \in \mathcal{I}(\operatorname{pre} p)} \mathcal{A}_i(\operatorname{pre} p) = \bigtimes_{i \in \mathcal{I}(\operatorname{pre} p)} \operatorname{proj}_{A_i} A(\operatorname{pre} p)$. In particular, this implies that $a \in A(\operatorname{pre} p)$. Hence, we have $\operatorname{pre} p \in V^t \subseteq V$, and $a \in A(\operatorname{pre} p)$, so that $(\operatorname{pre} p, a) \in V$ – specifically, $(\operatorname{pre} p, a) \in V^{t+1}$.

Conversely, fix $v = (\operatorname{pre} v, a) \in V^{t+1}$. We have that $\operatorname{pre} v \in V^t = P^t$. Moreover, by condition (APF-G) of Definition 8, we have $A(\operatorname{pre} v) = \bigtimes_{i \in \mathcal{I}(\operatorname{pre} v)} \operatorname{proj}_{A_i} A(\operatorname{pre} v)$. Lastly, note that $\mathcal{I}(\operatorname{pre} v) = \mathcal{D}(a) = \mathcal{D}(\tilde{f}(\operatorname{pre} v))$, where the first equality follows from condition (APM) of Definition 8 and the second one from the definition of $\tilde{f}$. Therefore, we conclude that $a \in \bigtimes_{i \in \mathcal{D}(\tilde{f}(\operatorname{pre} v))} \operatorname{proj}_{A_i} A(\operatorname{pre} v) = \bigtimes_{i \in \mathcal{D}(\tilde{f}(\operatorname{pre} v))} \mathcal{A}_i(\operatorname{pre} p)$, with the second equality following from the definition of $(\mathcal{A}_i)_{i \in I}$. As a result, we obtain that $v = (\operatorname{pre} v, a) \in \{\operatorname{pre} v\} \times \left(\bigtimes_{i \in \mathcal{D}(\tilde{f}(\operatorname{pre} v))} \mathcal{A}_i(\operatorname{pre} p)\right) \subseteq P$, where the inclusion holds by the recursive definition of set P (cf. Section 6.1). In particular, $p \in P^{t+1}$, and this concludes the induction.

Step 3: verify that, for each player, distinct plays are always distinguishable. Note that the feedback function $\tilde{f}$ is injective. Hence, for each $p, q \in P = V$ with $p \neq q$, $F_i(p) \neq F_i(q)$. ■

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