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Sources of Uncertainty and Subjective Prices*

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Abstract

We develop a general framework to study source-dependent preferences in economic contexts. We behaviorally identify two key features. First, we drop the assumption of uniform uncertainty attitudes and allow for *source-dependent attitudes*. Second, we introduce *subjective prices* to compare outcomes across different sources. Our model evaluates profiles source-wise, by computing the source-dependent certainty equivalents; the latter are converted into the unit of account of a common source and then aggregated into a unique evaluation. By viewing time and location as instances of sources, we show that subjective discount factors and subjective exchange rates are emblematic examples of subjective prices. Finally, we use the model to explore the implications on optimal portfolio allocations and home bias.

Key words: source preference, source-dependent uncertainty attitudes, subjective prices, competence hypothesis, home bias

1 Introduction

1.1 Reductionisms

In applications the consequences of different courses of action are often summarized through basic quantitative indicators, like amounts of money or casualties. Though convenient, these succinct expressions are reduced forms of genuine, but often complicated and elusive to come by, consequences that record "anything that may happen to the person" as Savage (1954, p. 13) prescribes. In the classic Savagean paradigm, utility functions are defined over such all inclusive consequences, and expected utilities are computed relative to a subjective probability on states of nature that captures agents' beliefs. Reduced-form

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consequences may give rise to problematic issues within this standard setting. As Smith (1969, p. 325) eloquently writes, "[...] if a man loses a dice game bet he and his associates might consider that he was merely the victim of bad luck [...]. But if he loses from incorrectly predicting a rise in the Dow-Jones, he may perceive that his colleagues feel that he should have known better. [...] the utility of money or other rewards is not independent of the circumstances under which it is obtained." A reduced-form consequence is, by its very nature, a crude description of all the relevant outcomes of a course of action. Altogether different consequences may, for example, end up being translated in the same amount of money, so in the same reduced-form consequence.[1]

This "consequence reductionism" impacts on another key feature of Savage's approach, that is, the reduction of uncertainty to risk: expected utilities are, effectively, computed with respect to the distributions on consequences (lotteries) induced by the subjective probability via the actions. Risk attitudes are determined with respect to these induced distributions, regardless of the underlying state spaces. As Smith's quote indicates, this "risk reductionism" becomes questionable when combined with reduced-form consequences that might well receive multiple evaluations according to the kind of contingencies that deliver them.

Source dependence We address this issue by extending the expected utility framework to accommodate *source-dependent preferences*. In line with the literature, a source is a collection of contingencies that correspond to the same "mechanism" of uncertainty; different sources correspond to different domains of uncertainty.[2] A paradigmatic example is the classic two-urn Ellsberg experiment where the two urns identify distinct sources that differ in the stochastic nature of the uncertainty faced.

Our framework allows us to go beyond this classic example by encompassing an array of factors that may account for the presence of multiple sources in a decision problem. First of all, sources may vary in the personal implications of the material consequences involved. For example, losing a certain amount of money can be accompanied by a sense of bad luck or of incompetence, depending on the attributes attached to such loss, as evoked by Smith. According to the *competence hypothesis* (Heath and Tversky, 1991, and Fox and Tversky, 1995), an individual may prefer gaining a certain amount of money in the domain of judgment rather than in the domain of chance if, in the former case, the gain is accompanied by a feeling of credit.[3] Source-dependent preferences may also arise in a

[1]Throughout the paper, the terms "consequence" and "outcome" are used interchangeably.

[2]Originally introduced by Tversky and Fox (1995) and Tversky and Wakker (1995), the notion of source preference is the object of study of more recent works including Chew and Sagi (2008), Abdellaoui, Baillon, Placido, and Wakker (2011) and Gul and Pesendorfer (2015).

[3]"Psychic payoffs of satisfaction or embarrassment can result from self-evaluation or from an evaluation by others. [...] In the domain of chance, both success and failure are attributed primarily to luck. The situation is different when a person bets on his or her judgment." (Heath and Tversky, 1991, pp. 7-8)

deterministic setting. For instance, the possibly distinct social implications of the same material consequence may be rationalized in terms of different (deterministic) sources. A policy maker may evaluate differently the threats to public security of different categories, even if they involve the same number of casualties. Slovic (1999, p. 691) puts this in clear terms: "Are the deaths of 50 passengers in separate automobile accidents equivalent to the deaths of 50 passengers in one airplane crash?" While the relevance of this type of queries for policy making is obvious, it begs the question of how one compares prospects that depend on different sources. Moreover, any answer involves inevitably a subjective assessment: different individuals would most likely provide different reasons as to whether or not the above two source-dependent outcomes are equivalent.[4]

Dates and locations Some insights about how to address source dependence may surprisingly come from seemingly unrelated contexts. Indeed, emblematic examples of sources that do not involve uncertainty are rooted in standard economic settings: in particular, sources can be identified by the different dates or locations at which a material consequence is delivered. The idea that time and location constitute formal attributes to be taken into account in the evaluation of a good dates back, at least, to Debreu (1959), who argues that "a commodity is a good or a service completely specified physically, temporally, and spatially." These classic analyses arm us with familiar tools to evaluate the same outcome at different dates or locations — namely, the discount factor and the exchange rate in temporal and allocation contexts, respectively. Importantly, these tools identify the subjective value of an outcome in relative terms. Thus, they can be referred to as the *subjective prices* of receiving a monetary outcome at a given date or location from the perspective of another date or location. We will see that subjective discount factors and subjective exchange rates are just two instances of a more encompassing notion of subjective price which lies at the core of our general theory of source dependence.

1.2 Source-dependent attitudes and subjective prices

Our goal is to develop a unifying framework for the study of source dependence. We provide an axiomatic foundation for a general model which shows that the multifaceted nature of source dependence can be reduced to two behavioral features that identify intra-source and inter-source tastes.

First, source dependence may originate from a multidimensional perception of uncertainty. The Ellsberg paradox is a clear example in which agents display source-dependent uncertainty attitudes. More generally, evidence from different contexts indicates that individuals may not exhibit uniform attitudes toward uncertainty.[5] We thus dispense with the

[4]Note that according to Savage, these outcomes would correspond to distinct consequences arising from different full descriptions.

[5]Such awareness has reached out policy makers, as suggested by the European Securities and Markets

standard assumption that there exist universal risk attitudes, portable across sources. We recognize, instead, that such attitudes may depend on the underlying source of uncertainty. Within the expected utility framework, this means that the von Neumann-Morgenstern utility functions are themselves allowed to be source dependent. Formally, we adopt the standard notion of certainty equivalent to identify the uncertainty attitude on each source. Consider a collection $I = \{1, \ldots, n\}$ of sources and let f_i stand for an uncertain prospect dependent on source $i \in I$. The certainty equivalent $c_i(f_i)$ corresponds to the monetary consequence that the agent finds equivalent to the uncertain prospect f_i on source i.[6] It identifies the *intra-source* tastes and can be interpreted as capturing the agent's degree of familiarity about the underlying source of uncertainty. The traditional approach continues to hold in comparing prospects that depend on the same source, but no longer otherwise.

This leads us to the second distinguishing feature of source dependence. After having factored out uncertainty through certainty equivalents, the evaluation of a deterministic outcome may still be source dependent. To compare actions that depend on different sources, we thus need to convert units on one source into units on another source via a subjective conversion rate. Such a rate can be viewed as a subjective price that identifies *inter-source* tastes by expressing the relative value of a deterministic consequence on one source in terms of its equivalent value on another source (for instance, reflecting agents' different competence on them).

More formally, given any prospects f_i and g_j on two sources i and j, our agent first computes the source-dependent certainty equivalents, $c_i(f_i)$ and $c_j(g_j)$, then applies a rate $\delta_{ij}(c_j(g_j))$ that converts $c_j(g_j)$ into the unit of account of source i. This rate is the subjective price of $c_j(g_j)$ on source i. Our agent prefers f_i to g_j if and only if $c_i(f_i) \geq \delta_{ij}(c_j(g_j))$. Note that in the two-urn Ellsberg experiment, the two sources differ only in terms of their stochastic nature while being perfectly comparable otherwise; in this case, the subjective price is given simply by the identity function. However, this does not hold in general, as suggested by Slovic (1999) in the context of casualties. Furthermore, our theory analyzes a richer domain of preferences which involves not only comparisons between source-dependent prospects, but also between streams of source-dependent prospects of the form $\mathbf{f} = (f_1, \ldots, f_n)$. According to our general model, a stream $\mathbf{f}$ is evaluated by converting each source-dependent certainty equivalent $c_i(f_i)$ into the unit of account of a common source o, and then by aggregating the o-normalized valuations $\delta_{oi}(c_i(f_i))$ as $W_o(\delta_{o1}(c_1(f_1)), \ldots, \delta_{on}(c_n(f_n)))$. We will focus on the prominent case of quasi-arithmetic aggregators and apply it to revisit both temporal and allocation settings.

Authority (May 2014): "We propose to clarify, with respect to clients' risk-bearing capacity, that any particular consumer has not only one and overall risk attitude but different risk attitudes towards different investment targets."

[6] Because of their economic relevance, throughout the paper we focus on monetary consequences, so on "monetary reductionism".

Temporal settings We will show that temporal decision problems can be viewed as instances of decision problems with source dependence, with standard criteria being characterized as special cases of our general model.[7] More specifically, $\mathbf{f} = (f_1, \ldots, f_n)$ can be interpreted as a temporal stream of consumption where source $i \in \{1, \ldots, n\}$ identifies the date at which bundle f_i is consumed. The temporal setting is an important special case of our theory for at least two reasons. First, it offers a classic example of conversion rate as given by the discount factor — the subjective price of consumption in the next period in terms of consumption in the current period. Second, the source-dependent certainty equivalents act as time-dependent utilities and give rise to time-dependent uncertainty attitudes. Thus, temporal sources suggest a behavioral basis to explain the empirical evidence that risk aversion may be domain-dependent and vary with the time horizon and with the age.[8] Recognizing such dependence may help explaining some puzzling findings in asset pricing.[9]

Allocation settings Evoking Debreu's (1959) ideas, standard allocation problems in international economics can also be viewed as instances of decision problems with source dependence: different locations at which the same material consequence can be delivered correspond to different sources. We apply our model to a simple portfolio allocation problem and show that our conversion rate plays the role of a subjective exchange rate that captures the subjective value of consumption on one market in terms of consumption on another market. We view it as the agent's relative degree of familiarity about the value of an outcome on different markets. Our result on optimal allocation uncovers a relationship between the subjective exchange rate and the (market) real exchange rate which is the spatial counterpart of the well-known relationship between the subjective discount factor and the real interest rate in temporal settings.

Moreover, source dependence may help explaining the large empirical evidence on under-diversification and home bias in capital markets.[10] Our theoretical results point to a recent empirical literature suggesting that one driving factor for under-diversification might be a familiarity bias: individual investors prefer local securities because they feel

[7]Note that the simple setup based on comparisons of outcome-source pairs encompasses the standard framework of time preferences in which agents compare reward-time pairs.

[8]Baucells and Heukamp (2010), Abdellaoui, Diecidue, and Onculer (2011), and Eisenbach and Schmalz (2016) report evidence on increasing risk aversion as the source of risk approaches in time. On age, see, e.g., Deakin, Aitken, Robbins, and Sahakian (2004), and Dohmen et al. (2011).

[9]For instance, contrary to the standard predictions, van Binsbergen, Brandt, and Koijen (2012), Andries, Eisenbach, Schmalz, and Wang (2015), Andries, Eisenbach, and Schmalz (2018), and Weber (2018) report evidence in favor of a downward-sloping term structure of equity returns.

[10]This phenomenon has been initially documented by French and Poterba (1991). Pointing to Heath and Tversky's competence hypothesis, they recognize that (p. 225) "Investors may not evaluate the risk of different investments based solely on the historical standard deviation of returns. They may impute extra "risk" to foreign investments because they know less about foreign markets, institutions, and firms." For a comprehensive survey, see, e.g., Campbell (2006).

more knowledgeable about nearby markets, even if this sense of competence cannot be attributed to superior information.[11] Note that this notion of familiarity bias is distinct from the preference for familiarity arising from effective knowledge about the underlying mechanism of uncertainty. The latter notion corresponds to Heath and Tversky's competence hypothesis and is controlled by the source-dependent ambiguity attitudes (captured by the certainty equivalents). Thus, our model accommodates both forces and allows us to analyze the impact of each of them on home bias.

Outline The rest of the paper is organized as follows. Next subsection discusses the related literature. The next section presents the formal setup. Section 3 states the basic axioms and representation results of preferences over prospect-source pairs. Section 4 contains our general characterization result of preferences over prospect profiles, as well as two prominent special cases: the well-known quasi-arithmetic specification and the temporal setting. Section 5 provides a comparative statics analysis of what it means for one agent to have a higher preference for source smoothing than another. Section 6 introduces source dependence into a standard allocation problem and studies the implications in terms of portfolio diversification and home bias. Section 7 concludes with some remarks on how one could extend our setup to allow for certain forms of interdependence between sources. All proofs are collected in appendix.

1.3 Related literature

There is by now a sizable body of theoretical and experimental works which recognize the dependence of choice behavior on sources of uncertainty. Klibanoff, Marinacci, and Mukerji (2005), Nau (2006), Ergin and Gul (2009), and Seo (2009) start the modern literature on source dependence by considering two "hierarchical" sources.[12] The focus has then moved toward nuanced comparisons involving sources for which agents exhibit different degrees of uncertainty. The closest works to the present paper are Chew and Sagi (2008), Abdellaoui, Baillon, Placido, and Wakker (2011), and Gul and Pesendorfer (2015). These papers identify a source by a restricted collection of events with respect to which the agent exhibits a uniform degree of ambiguity attitude. The key novelty is the recognition that preferences can be compatible with a single probability distribution as long as acts depend on the same source; thus, probabilistic sophistication holds within each source but not necessarily across sources. By suitably extending de Finetti's notion of exchangeability, Chew and Sagi (2008) provide sufficient axiomatic conditions for prob-

[11] Among others, Grinblatt and Keloharju (2001), Huberman (2001), Zhu (2002), Goetzmann and Kumar (2008), Graham, Harvey, and Huang (2009), Seasholes and Zhu (2010), and Boyle, Garlappi, Uppal, and Wang (2012) find that familiarity is unlikely to be driven by access to superior information about local markets.

[12] See Marinacci (2015) for an analysis of two-stage decision models.

abilistic sophistication to hold locally, within each source referred to as a "small world." Abdellaoui et al. test for the empirical relevance of source dependence by developing an experimental method which assumes probabilistic sophistication on each source, and it is based on the estimate of three parameters: a utility index, a subjective probability over states, and a source function which transforms probabilities into decision weights. As in Chew and Sagi, only the latter is source dependent. They find robust experimental evidence in support of source-dependent uncertainty attitudes in different decision problems. Gul and Pesendorfer (2015) characterize a special form of α-maxmin preferences which allows for source dependence. A source is subjectively identified by a non-atomic probability measure over a restricted σ-algebra of events relative to which probabilistic sophistication holds. Similarly to Abdellaoui et al., the act-induced lotteries are evaluated according to rank-dependent expected utility with a source-dependent probability transformation function identifying the uncertainty attitude toward specific sources.

In addition to Abdellaoui et al., the above theoretical findings are supported by substantial empirical evidence pointing to domain-dependent uncertainty attitudes. In particular, there has been a growing interest in studying not only attitudes toward artificial sources, such as the two Ellsberg urns, but also toward natural sources (e.g., the temperature in different cities or different stock indices).[13]

The present paper contributes to the above literature in two main directions. First, we introduce a focal element given by the notion of subjective price. Acting as rates of conversion, subjective prices allow us to compare source-dependent prospects as well as entire streams of source-dependent prospects. Moreover, they offer a novel behavioral perspective to study some classic applied settings and related empirical findings.

Second, our model accommodates not only source-dependent probability distortions (i.e., weighting of probabilities that depends on sources) but also source-dependent utilities (i.e., weighting of consequences that depends on sources). While the aforementioned works adopt the former approach, the latter one is in line with some early discussions of the Ellsberg paradox (notably, Roberts, 1963, and Smith, 1969), and the literature on domain dependence of risk aversion. Our model thus incorporates alternative approaches by putting them into perspective.

2 Setup

The basic elements of a decision problem with source dependence are a set X of *outcomes* and a collection I of *sources of uncertainty* which affect the outcomes resulting from the agent's choices.

[13]See, e.g., Fox and Tversky (1998), Kilka and Weber (2001), Weber, Blais, and Betz (2002), Baillon and Bleichrodt (2015), Abdellaoui, Bleichrodt, Kemel, and l'Haridon (2017), Baillon, Huang, Selim, and Wakker (2018), and Li, Müller, Wakker, and Wang (2018).

We assume that X is a non-singleton interval of the real line containing the null outcome 0. In particular, having $X = \mathbb{R}$ is natural when outcomes are monetary payoffs. On the other hand, if they represent quantities of a homogeneous good (or casualties), it is natural to allow only for non-negative real numbers. Note that here outcomes are state-independent. We thus abstract from state-dependence, a classic issue in decision making under uncertainty (see Karni, 1985, and Dreze, 1987). On the other hand, our setup can be viewed as a "space-dependent" one – where the spaces are given by the S_i's or the Σ_i's described below.

For simplicity, let $I = \{1, \ldots, n\}$ be a finite set. A source of uncertainty $i \in I$ is a collection of contingencies that correspond to a common mechanism of uncertainty. A source-dependent prospect describes how these contingencies affect the outcomes of a chosen alternative. Next, we present two ways of modeling sources (and, therefore, source-dependent prospects) according to how such contingencies are formalized.

2.1 Sources and source-dependent prospects

Naive definitions As a first way of describing sources of uncertainty, consider a collection $\{S_i\}_{i \in I}$ of finite sets of contingencies. Each source $i \in I$ identifies a set S_i, with generic element denoted by s_i, that can be thought of as the set of contingencies which features the common source of uncertainty i. A *prospect dependent on source* i is a function $f_i : S_i \to X$ which yields outcome $f_i(s_i) \in X$ if contingency s_i occurs. We denote by $\mathcal{F}_i := X^{S_i}$ the set of all prospects that depend on source i. When no confusion arises, we denote by x both the outcome x in X (a scalar) and the constant source-dependent prospect $x1_{S_i} \in \mathcal{F}_i$ (a function) yielding x in every contingency in S_i.

A *prospect profile* is a vector $\mathbf{f} = (f_1, \ldots, f_n)$, or shortly $\mathbf{f} = (f_i)_{i \in I}$, that specifies for each source i the corresponding source-dependent prospect f_i. Let $\mathcal{F} := \times_{i \in I} \mathcal{F}_i$ denote the set of all prospect profiles. With little abuse, we denote by $\mathbf{x}$ both the outcome profile $\mathbf{x} = (x_i)_{i \in I} \in X^I$ and the corresponding profile $(x_i 1_{S_i})_{i \in I}$ of constant prospects in $\mathcal{F}$.

For example, in the Ellsberg two-urn context, $I = \{1, 2\}$ is the set of urns, $S_i = \{\text{Red}_i, \text{Black}_i\}$ is the set of contingencies for each $i \in I$, and a prospect profile is a "portfolio" of bets

$$\mathbf{f} = ((\text{Red}_1, x_1; \text{Black}_1, y_1), (\text{Red}_2, x_2; \text{Black}_2, y_2))$$

that describes the payoffs of an agent who (potentially) gambles on both urns.

If $f_j = 0$ for all $j \neq i$, then the prospect profile $\mathbf{f}$ depends only on source i and it is denoted by (f_i, i). Let $\mathcal{P}_i := \{(f_i, i) : f_i \in \mathcal{F}_i\}$ be the family of all these prospects. Thus, $\mathcal{P} = \cup_{i \in I} \mathcal{P}_i$ is the family of all prospects that may yield non-zero outcomes on only one source at a time; we refer to these prospects as *prospect-source pairs*. For example, in the Ellsberg two-urn context, (f_i, i) corresponds to a bet f_i on urn i, and the famous paradox involves comparisons of bets that depend only on one urn at a time.

Agent's preferences are described by a binary relation $\succsim$ on the set $\mathcal{F}$ of all prospect profiles. Throughout, we will often focus on the restriction of $\succsim$ to the smaller set $\mathcal{P}$ of all prospect-source pairs. In particular, the further restriction of $\succsim$ to $\mathcal{P}_i$ induces a preference $\succsim_i$ on $\mathcal{F}_i$ defined by

$$f_i \succsim_i g_i \iff (f_i, i) \succsim (g_i, i) \quad \text{for all } i \in I.$$

Sophisticated definitions As a second way of describing sources of uncertainty, consider a collection $\{\Sigma_i\}_{i \in I}$ of finite algebras on a given state space S.[14] Each source i corresponds to an algebra Σ_i of subsets of S. In turn, the atoms of each algebra can be thought of as the set of contingencies featured by a given source. A prospect dependent on source i is now a Σ_i-measurable function $f_i : S \to X$ that yields outcome $f_i(s) \in X$ if state s occurs. We denote by $\mathcal{F}_i$ the collection of all these prospects. The definitions of $\mathcal{F}$, (f_i, i), $\mathcal{P}_i$, and $\mathcal{P}$ are formally unchanged.[15]

The relation with the previous framework is straightforward: it is sufficient to set $S = \times_{i \in I} S_i$, consider the algebras

$$\Sigma_i = \{A_i \times S_{-i} : A_i \subseteq S_i\} \quad \text{for all } i \in I,$$

and observe that the value $f_i(s)$ of an element of $\mathcal{F}_i$ depends only on the i-th component s_i of s.

Throughout we adopt this framework since it is more general than the naive one and preserves tractability. However, our axiomatic derivation and related results do not depend on the specific formulation adopted.

3 Preferences and basic representation results

We refer to $\succsim_i$ on $\mathcal{F}_i$ for $i \in I$ as *intra-source preferences* and to $\succsim$ on $\mathcal{P}$ as *inter-source preferences*. The entire relation $\succsim$ on $\mathcal{F}$ describes *preferences over prospect profiles*.

3.1 Intra-source preferences

We first study preferences within sources. For every source i, we impose the following basic assumption on $\succsim_i$ over $\mathcal{F}_i$.

Axiom A. 1 *For every $i \in I$ and $f_i, g_i, h_i \in \mathcal{F}_i$,*

(i) if $f_i \geq g_i$, then $f_i \succsim_i g_i$. Moreover, if both f_i and g_i are constant and $f_i > g_i$, then $f_i \succ_i g_i$;

[14] We consider finite algebras for convenience. Our results can be extended to infinite algebras.

[15] This approach is adopted by Chew and Sagi (2008), Abdellaoui et al. (2011), and Gul and Pesendorfer (2015).

(ii) if $f_i \succ_i g_i \succ_i h_i$, then there exist $\alpha, \beta \in (0,1)$ such that

$$(1-\alpha) f_i + \alpha h_i \succ_i g_i \succ_i (1-\beta) f_i + \beta h_i.$$

These properties have an obvious interpretation since only prospects depending on a given source i are considered. Condition (i) is a standard monotonicity assumption stating that the agent should prefer a source-dependent prospect that dominates another in every state. Moreover, the converse holds, as well, if the prospects at hand are constant. Condition (ii) is a standard Archimedean property that guarantees the continuity of each preference $\succsim_i$.

Definition 1 *For every $i \in I$, a monotone and continuous functional $c_i : \mathcal{F}_i \to X$ is a* ***certainty equivalent*** *for $\succsim_i$ if, for all $x \in X$, $x = c_i(x)$ and, for all $f_i \in \mathcal{F}_i$,*

$$f_i \sim_i c_i(f_i).$$

The following routine lemma shows that Axiom A.1 guarantees the existence of a unique certainty equivalent.

Lemma 2 *For every $i \in I$, let $\succsim_i$ be a binary relation on $\mathcal{F}_i$. The following conditions are equivalent:*

(i) $\succsim_i$ is a weak order that satisfies Axiom A.1;

(ii) there exists a unique certainty equivalent functional $c_i : \mathcal{F}_i \to X$ for $\succsim_i$. In this case, for all $f_i, g_i \in \mathcal{F}_i$,

$$f_i \succsim_i g_i \iff c_i(f_i) \geq c_i(f_i).$$

Certainty equivalents play a key role in our analysis of source dependence. Not only they provide a utility representation of intra-source preferences; more importantly, they allow us to identify the agent's attitude toward uncertainty on each source. As argued earlier, we can distinguish between source-dependent probabilities — consistent with the evidence on non-uniform ambiguity attitudes — and source-dependent utilities — consistent with the evidence on the multidimensionality of risk. The following classic examples illustrate this point by presenting alternative functional specifications.

Example 3 (Source-dependent expected utility) *For every $i \in I$, there are a strictly increasing and continuous function $u_i : X \to \mathbb{R}$ and a probability measure p_i on (S, Σ_i) such that, for all $f_i \in \mathcal{F}_i$,*

$$c_i(f_i) = u_i^{-1}\left(\mathbb{E}_{p_i}[u_i(f_i)]\right). \tag{1}$$

In the above expression, $c_i(f_i)$ is the u_i-mean of f_i with respect to p_i. It corresponds to a basic example of a Chisini mean (1929, p. 113). More precisely, the means of this kind are called *quasi-linear* Chisini mean[16] and their characterization dates back to de Finetti (1931). Axiom A.2 guarantees the existence of a certainty equivalent functional as in (1). There exist different ways of expressing this standard assumption in terms of binary relations. As they are all well known, we omit their formal exposition and refer to the literature.[17]

Axiom A. 2 *For every $i \in I$, there exists a quasi-linear Chisini mean $c_i : \mathcal{F}_i \to X$ that represents $\succsim_i$.*

The above axiom amounts to say precisely that the agent is an expected utility maximizer source-by-source.

A popular generalization of (1) considers a source function w_i that transforms probability measures into decision weights. This approach gives rise to a rank-dependent formulation of source dependence in the same spirit as Abdellaoui et al. (2011).

Example 4 (Source-dependent probability-weighting) *For every $i \in I$, there are a probability measure p_i on (S, Σ_i) and an increasing and onto function $w_i : [0,1] \to [0,1]$ such that, for all $f_i \in \mathcal{F}_i$,*

$$c_i(f_i) = u^{-1}\left(\mathbb{E}_{w_i(p_i)}[u(f_i)]\right), \tag{2}$$

where $u : X \to \mathbb{R}$ is a source-independent strictly increasing and continuous function.

Naturally, many other examples are possible: all is needed, up to now, for our theory of source dependence is the existence of certainty equivalent functionals to represent intra-source preferences. Thus, any model that admits certainty equivalents can be adopted to specify the functional form of c_i. For instance, one could use models of ambiguity-sensitive behavior. Below we suggest a source-dependent version of rational preferences under ambiguity (Cerreia-Vioglio et al., 2011).

Example 5 (Source-dependent preferences under ambiguity) *For every $i \in I$, there are a strictly increasing and continuous function $u_i : X \to \mathbb{R}$ and a monotone, normalized, and continuous function $I_i : u_i(X) \to \mathbb{R}$ such that, for all $f_i \in \mathcal{F}_i$,*

$$c_i(f_i) = u_i^{-1}\left(I_i(u_i(f_i))\right). \tag{3}$$

[16]Appendix A studies the link between the notion of Chisini mean and the preference representations proposed in this paper. See, e.g., Bullen (2003) and Grabisch, Marichal, Mesiar, and Pap (2011) for a modern treatment of Chisini means.

[17]See, e.g., Savage (1954) and Wakker (1988, 1989) for infinite and finite algebras Σ, respectively.

This general structure encompasses various models of decision-making under ambiguity. For instance, a special case of (3) is given by the following source-dependent specification of the maxmin model (Gilboa and Schmeidler, 1989):

$$c_i(f_i) = \min_{p_i \in C_i} u_i^{-1}\left(\mathbb{E}_{p_i}[u_i(f_i)]\right),$$

where C_i is a compact set of probabilities. According to this criterion, the agent can exhibit different attitudes toward ambiguity depending on sources. For instance, he can be more ambiguity averse on one source than on another;[18] he may as well exhibit ambiguity neutrality on some source i if the set C_i consists of a single probability.

Our approach allows also to study source dependence in models that focus on non-expected utility features that are distinct from ambiguity. For instance, if we replace the utility u_i in Example 3 with a compact set of utilities U_i, we obtain the following source-dependent specification of the cautious expected utility model (Cerreia-Vioglio, Dillenberger, and Ortoleva, 2015):

$$c_i(f_i) = \min_{u_i \in U_i} u_i^{-1}\left(\mathbb{E}_{p_i}[u_i(f_i)]\right).$$

The multiplicity of utilities reflects agent's uncertainty about his risk attitudes within each source. The relevant aspect in the above formulation lies in the heterogeneity of the utility sets across sources, reflecting both source-dependent risk attitudes as well as uncertainty about his own risk attitude on each source. An individual might be fully confident in a single utility function for a specific source, thereby behaving as a standard expected utility maximizer over that source.

3.2 Inter-source preferences

Before turning to the overall relation $\succsim$ on $\mathcal{F}$, we focus on its restriction to the space $\mathcal{P} := \{(f_i, i) : f_i \in \mathcal{F}_i\}$. This subsection provides a characterization of preferences between prospect-source pairs in $\mathcal{P}$. Next axiom is the key in relating monetary outcomes generated by distinct sources.

Axiom B. 1 *(Inter-source tastes) For every $i, j \in I$ and $x \in X$, there exists $y \in X$ such that $(x, i) \sim (y, j)$.*

This is a comparability assumption across sources. It requires that, for any two sources i and j, and any sure outcome x generated by source i, the agent is always able to identify a sure outcome y generated by source j which is equivalent to x. For example, a policy maker may evaluate differently the same number of (sure) casualties due to a terror attack

[18] By applying the comparative notion of Ghirardato and Marinacci (2002), we can say that an agent is more ambiguity averse on source i than on source j if $C_i \supseteq C_j$, provided that u_i and u_j are cardinally equivalent.

or to a seasonal flu: he may regard, say, 100 sure casualties from the first source as having the same impact as $1,000$ sure casualties from the second (see, e.g., Slovic, 1999, for similar examples). In terms of the representation, these inter-source comparisons will be captured by functions dubbed *rates*, as defined next.

Definition 6 *A self-map $\delta_{ij} : X \to X$ is a **rate** if it is strictly increasing, with $\delta_{ik} = \delta_{ij} \circ \delta_{jk}$ and $\delta_{ii} = \mathrm{id}_X$, for all $i, j, k \in I$.*

For any $x \in X$, the rate $\delta_{ij}(x) \in X$ identifies the outcome on source i that makes the agent indifferent to outcome x on source j. With this notion, we can now state our behavioral characterization of preferences over prospect-source pairs.

Proposition 7 *Let $\succsim$ be a binary relation on $\mathcal{P}$. The following conditions are equivalent:*

- *(i) $\succsim$ is a weak order that satisfies Axioms A.1 and B.1;*
- *(ii) there exist*
  - *(a) a family $(c_i)_{i \in I}$ of **certainty equivalents** for $(\succsim_i)_{i \in I}$,*
  - *(b) a family $(\delta_{ij})_{i,j \in I}$ of **rates**,*

  such that, for all $i, j \in I$, $f_i \in \mathcal{F}_i$ and $g_j \in \mathcal{F}_j$,

$$(f_i, i) \succsim (g_j, j) \iff c_i(f_i) \geq \delta_{ij}(c_j(g_j)). \tag{4}$$

Moreover, the elements of $(c_i)_{i \in I}$ and $(\delta_{ij})_{i,j \in I}$ are unique.

Representation (4) suggests a two-step procedure to compare prospect-source pairs. First, the pairs (f_i, i) and (g_j, j) are independently evaluated by computing their certainty equivalents, $c_i(f_i)$ and $c_j(g_j)$, respectively. For every source i, the function c_i can be viewed as a unit of account for that source. Second, the certainty-equivalent evaluations are converted into the unit of account of one source. Such conversion is carried out by the function δ_{ij} which reflects the rate of substitution of outcomes between sources i and j. In particular, representation (4) guarantees that, for any $x, y \in X$ and $i, j \in I$,

$$(x, i) \sim (y, j) \iff x = \delta_{ij}(y) \iff y = \delta_{ji}(x).$$

Hence, δ_{ij} plays precisely the role of an inter-source rate of substitution. Moreover, observe that preferences over prospect-source pairs are uniquely determined by the source-dependent certainty-equivalents and the rates. In turn, both of them can be easily elicited from choice behavior (e.g., experimental data).

The essential contribution of Proposition 7 lies in recognizing that source dependence requires a distinction between intra-source and inter-source tastes. Intra-source tastes correspond to a standard notion of preferences under uncertainty and are easily determined because they are monotone within each source (and $X \subseteq \mathbb{R}$). The notion of inter-source tastes, instead, is novel and lies at the core of our model. It arises from the fact that different sources affect the desirability of outcomes, so that an individual may prefer receiving outcome x on source i rather than the same outcome on another source j. Representation (4) states that inter-source tastes are identified by the rates $(\delta_{ij})_{i,j}$ which provide a measure of how the agent values a certain outcome on a certain source in terms of an equivalent outcome on a different source. Thus, $(\delta_{ij})_{i,j}$ play the role of subjective prices.

Finally, note that a source-dependent certainty equivalent depends only on the uncertainty attitude on that source because intra-source tastes are monotone. Therefore, representation (4) provides a separation between inter-source tastes and attitudes toward uncertainty (including risk and ambiguity).

3.2.1 The expected utility case

An interesting special case occurs when the sure outcome x is on source j equivalent to itself on source i.

Axiom B. 2 *(Pecunia non olet) For every $i, j \in I$ and $x \in X$, $(x, i) \sim (x, j)$.*

This axiom requires that, when deterministic outcomes are concerned, the agent does not discriminate between the sources generating them. Any difference in the evaluations of uncertain prospects must be due solely to the agent holding different uncertainty attitudes toward different sources. That is, the only perceived difference between sources is the quality of the information about the stochastic nature of the states of the world.

When monetary payoffs are involved, the fact that deterministic outcomes are unaffected by the stochastic nature of the underlying source rules out any consideration about the agent's competence, thus making this assumption particularly transparent. Its behavioral content can be summarized by the time-honored claim of Vespasian that *pecunia non olet* — money has no smell. For instance, the value of a monetary outcome is the same on both Ellsberg urns, as shown below. That said, one can think of situations in which this assumption would be too restrictive: it is likely that a person would not be indifferent between receiving the same monetary profit from legal versus illegal entrepreneurial activities.[19]

Replacing Axiom B.1 with Axiom B.2 in Proposition 7 is immediately seen to correspond to $\delta_{ij} = \mathrm{id}_X$ for all $i, j \in I$. Then, representation (4) reduces to

$$(f_i, i) \succsim (g_j, j) \iff c_i(f_i) \geq c_j(g_j). \tag{5}$$

[19] As mentioned earlier, Slovic (1999) contains various examples, mostly in the context of casualties.

In view of the importance of expected utility, we next provide a special case of Proposition 7 in which certainty equivalents can be directly compared – without the mediation of the rates δ_{ij} – and, at the same time, they conform to subjective expected utility.

Proposition 8 *Let $\succsim$ be a binary relation on $\mathcal{P}$. The following conditions are equivalent:*

(i) $\succsim$ is a weak order that satisfies Axioms A.2 and B.2;

(ii) there exist

(a) a family $(u_i)_{i\in I}$ of strictly increasing and continuous functions $u_i : X \to \mathbb{R}$,

(b) a family $(p_i)_{i\in I}$ of probability measures p_i on Σ_i,

such that, for all $i, j \in I$, $f_i \in \mathcal{F}_i$ and $g_j \in \mathcal{F}_j$,

$$(f_i, i) \succsim (g_j, j) \iff u_i^{-1}\left(\int_S u_i(f_i)\, dp_i\right) \geq u_j^{-1}\left(\int_S u_j(g_j)\, dp_j\right). \tag{6}$$

Moreover, the elements of $(u_i)_{i\in I}$ are cardinally unique and those of $(p_i)_{i\in I}$ are unique, provided that, for each $i \in I$, there are at least two non-null states in S_i.[20]

According to representation (6), source dependence manifests only in the potentially different uncertainty attitudes, whereas the value of a deterministic outcome is source independent. This is an important special case of our model as it offers a formal treatment of the utility approach to source dependence in alternative to the more traditional probability-weighting approach. We next illustrate the differences between these two approaches by applying them to resolve the classic two-urn Ellsberg paradox that originally sparked the economic interest on source-dependent preferences.

Two-urn Ellsberg experiment As anticipated in Section 2, we denote by $I = \{1, 2\}$ the set of urns and by $S_i = \{\text{Red}_i, \text{Black}_i\}$ the set of states for each $i \in I$. Urn 1 contains 50 red and 50 black balls, while Urn 2 has unknown composition. Denote by f_i (resp., g_i) the bet that pays \$100 if the ball drawn from urn $i \in I$ is red (resp., black) and \$0 otherwise. The typical pattern is $f_1 \sim g_1 \succ g_2 \sim f_2$. Most subjects are indifferent between betting on red or on black from either urn; yet, they strictly prefer to bet on Urn 1 rather than on Urn 2.

Source preferences generate a 'paradox' because indifference between betting on red or on black from either urn suggests that the two states are perceived as equally likely; but, if subjects assign also the same utilities to the same consequences generated by different

[20] As usual, a state $s \in S_i$ is null if indifference holds between any two profiles that differ only in state s, that is, if $x\{s\}f_i \sim_i y\{s\}f_i$ for all $f_i \in \mathcal{F}_i$ and $x, y \in X$.

sources, then the certainty equivalents of all the four bets must be the same. This is clearly inconsistent with a preference for betting on the known urn.

The source-dependent utility criterion developed in Proposition 8 can rationalize the observed pattern by maintaining the same equal probability of the states, while allowing for utilities to depend on sources as follows:

$$c_1(f_1) = u_1^{-1}\left[\frac{1}{2}u_1(100) + \frac{1}{2}u_1(0)\right] \neq u_2^{-1}\left[\frac{1}{2}u_2(100) + \frac{1}{2}u_2(0)\right] = c_2(f_2),$$

explaining the pattern if u_1 is less concave than u_2. Early discussions of this approach appear in Fellner (1961), Roberts (1963), Smith (1969), and, more recently, in Chew, Li, Chark, and Zhong (2008). On the other hand, the dominant approach in the literature follows the seminal papers of Kahneman and Tversky (1979) and Schmeidler (1989) by focussing on source-dependent *probability distortions* as follows:

$$\begin{aligned} c_1(f_1) &= u^{-1}\left[w_1\left(\frac{1}{2}\right)u(100) + w_1\left(\frac{1}{2}\right)u(0)\right] \\ &\neq u^{-1}\left[w_2\left(\frac{1}{2}\right)u(100) + w_2\left(\frac{1}{2}\right)u(0)\right] = c_2(f_2), \end{aligned}$$

explaining the pattern if $w_1 > w_2$.

3.3 Preferences over prospect profiles

It remains to study the preferences $\succsim$ over the entire domain $\mathcal{F}$ of prospect profiles. We assume that they satisfy the following basic axiom.

Axiom C. 1 *The binary relation $\succsim$ on $\mathcal{F}$ is a weak order such that:*

(i) (Monotonicity) if $\mathbf{f}, \mathbf{g} \in \mathcal{F}$ and $f_i \succsim_i g_i$ for all $i \in I$, then $\mathbf{f} \succsim \mathbf{g}$;

(ii) (Strict Monotonicity) if both $\mathbf{x}$ and $\mathbf{y}$ in X^I are constant and $\mathbf{x} > \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$;

(iii) (Archimedean Continuity) if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^I$ and $\mathbf{x} \succ \mathbf{y} \succ \mathbf{z}$, then there exist $\alpha, \beta \in (0,1)$ such that

$$(1-\alpha)\mathbf{x} + \alpha\mathbf{z} \succ \mathbf{y} \succ (1-\beta)\mathbf{x} + \beta\boldsymbol{z}.$$

Axiom C.1 can be viewed as the analogue of Axiom A.1 over the richer domain of prospect profiles. Properties (i) and (ii) are monotonicity assumptions. Property (i) maintains that the agent should prefer a profile $\mathbf{f}$ to another profile $\mathbf{g}$ whenever $\mathbf{f}$ consists of better source-dependent prospects than $\mathbf{g}$, source-by-source. Property (ii) is a form of strict monotonicity imposed on fully constant prospect profiles. Finally, property (iii) is a standard continuity property which, together with the other assumptions in Axiom C.1, guarantees that, for any $\mathbf{f} \in \mathcal{F}$, the agent can find a fully constant profile $\mathbf{x}$ which is indifferent to $\mathbf{f}$.

4 Representation results

This section provides behavioral characterizations of source-dependent preferences over prospect profiles — the domain of central interest of our theory. We start with a general utility structure that captures the essential features of source-dependent preferences.

4.1 General representation result

The following terminology will be useful to state our results.

Definition 9 *A monotone and continuous functional $D : X^I \to X$ is*

(i) an ***aggregator*** *if it is normalized, that is, $D(x1_I) = x$ for all $x \in X$;*

(ii) an ***aggregator at the numeraire*** $\mathbf{i} \in I$ *if it is i-normalized, that is, $D(x1_i) = x$ for all $x \in X$.*

We are now ready to state our general representation result. It disentangles the two behavioral features that may give rise to source dependence, namely intra-source and inter-source tastes, within the rich domain of prospect profiles.

Theorem 10 *Let $\succsim$ be a binary relation on $\mathcal{F}$. The following conditions are equivalent:*

(i) $\succsim$ satisfies Axioms A.1, B.1, and C.1;

(ii) there exist

(a) a family $(c_i)_{i \in I}$ of ***certainty equivalents*** *for $(\succsim_i)_{i \in I}$,*

(b) a family $(\delta_{ij})_{i,j \in I}$ of ***rates****,*

such that, for all $i, j \in I$, $f_i \in \mathcal{F}_i$, and $g_j \in \mathcal{F}_j$,

$$(f_i, i) \succsim (g_j, j) \iff c_i (f_i) \geq \delta_{ij} (c_j (g_j)),$$

(c) ***an aggregator at the numeraire*** $\mathbf{o} \in I$*, $W_o : X^I \to X$, such that, for every $\mathbf{f} = (f_i)_{i \in I}$,*

$$V_o (\mathbf{f}) = W_o (\delta_{o1} (c_1 (f_1)), \ldots, \delta_{on} (c_n (f_n))) \tag{7}$$

represents $\succsim$ on $\mathcal{F}$.

According to representation (7), a prospect profile is first evaluated source-wise, with each source i's own unit of account given by the certainty equivalent functional c_i. Each of these evaluations is then converted into the unit of account of a posited numeraire source o by applying the conversion rates δ_{oi}. Finally, this profile of evaluations is aggregated in a unique evaluation given by W_o which preserves the normalization to source o. Thus, W_o is an aggregator of source-dependent deterministic payoffs evaluated through the lens of the numeraire o.[21]

By Theorem 10, we can represent a binary relation $\succsim$ on $\mathcal{F}$ that satisfies Axioms A.1, B.1, and C.1 by a triple $((c_i)_{i\in I}, (\delta_{ij})_{i,j\in I}, W_o)$. We will refer to such a triple as a *source-dependent representation* of $\succsim$. Our utility representation has standard uniqueness properties, as shown next.

Proposition 11 *Let $\succsim$ be a binary relation on $\mathcal{F}$ that satisfies the assumptions of Theorem 10. Then, the following statements are true:*

(i) for every $o \in I$, the source-dependent representation $((c_i)_{i\in I}, (\delta_{ij})_{i,j\in I}, W_o)$ of $\succsim$ is unique;

(ii) if $o \neq \bar{o}$ and $((c_i)_{i\in I}, (\delta_{ij})_{i,j\in I}, W_o)$ and $((c_i)_{i\in I}, (\delta_{ij})_{i,j\in I}, W_{\bar{o}})$ are two source-dependent representations of $\succsim$, then $W_o = \delta_{o\bar{o}}(W_{\bar{o}})$.

Theorem 10 serves mainly the purpose of providing a general framework for the analysis of source dependence. It encompasses numerous special cases which vary in the functional specifications of the aggregator W_o and the rates δ_{ij} (as well as the forms of the certainty equivalents, as illustrated in Examples 3-5). For instance, one could think of non-additive aggregators (in the spirit of Choquet expected utility (Schmeidler, 1989) or maxmin preferences (Gilboa and Schmeidler, 1989) to introduce considerations of ambiguity in the process of evaluating sources. In what follows, we turn attention to important special cases.

4.2 Quasi-arithmetic aggregators

Theorem 10 develops an axiomatic foundation that captures the core features of source-dependent preferences over prospect profiles. A set of mild axioms is shown to be equivalent to a utility representation characterized by a minimal number of uniquely derived parameters. In this respect, it offers a flexible setting to study source dependence and understand its driving forces. On the other hand, the theorem does not suggest an operational criterion to adopt in applied decision problems. Indeed, the aggregator W_o, as well as the certainty equivalents, can take different functional forms, as previously argued. We address this issue by recognizing that W_o can be viewed as a mean. Then, one natural possibility consists of letting W_o be an arithmetic mean of the source-dependent certainty equivalents.[22] This section considers a slightly more structured specification and

[21] In Appendix B.2, Theorem 22 provides an alternative utility representation according to which the agent evaluates prospect profiles without committing to a specific numeraire.

[22] This specification is typical of multi-criteria decision analyses.

focuses on the well-known special case of quasi-arithmetic means. We will see that the class of quasi-arithmetic aggregators is analytically tractable and accommodates classic utility specifications used in applied economic settings.

In the remaining results of this section, we assume that $X = \mathbb{R}$ and impose few additional properties on $\succsim$ restricted to the domain of constant prospect profiles. We begin with a standard continuity assumption.

Axiom D. 1 *(Continuity) For each* $\mathbf{x} \in X^I$*, the sets*

$$\left\{\mathbf{y} \in X^I : \mathbf{y} \succsim \mathbf{x}\right\} \text{ and } \left\{\mathbf{y} \in X^I : \mathbf{x} \succsim \mathbf{y}\right\}$$

are closed.

Next axiom is a classic consistency property. It is a form of separability (in particular, a strengthening of the standard Coordinate Independence property) that is needed to obtain the desired quasi-arithmetic aggregator.[23]

Axiom D. 2 *(Tradeoff consistency) For each* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in X^I$*, each* $x, y, z, w \in X$*, and each* $i, j \in I$*, the condition*

$$\boldsymbol{b}_{-i}y \succsim \boldsymbol{a}_{-i}x \quad \boldsymbol{a}_{-i}z \succsim \boldsymbol{b}_{-i}w \quad \boldsymbol{c}_{-j}x \succsim \boldsymbol{d}_{-j}y$$

implies $\boldsymbol{c}_{-j}z \succsim \boldsymbol{d}_{-j}w$*.*

Next proposition shows that Axioms D.1 and D.2, together with the basic properties of the general theorem, deliver the desired representation. The proposition thus extends the classic quasi-arithmetic representation by introducing source dependence and, therefore, recognizing the multidimensionality of risk attitudes observed in the data.

Proposition 12 *Let* $\succsim$ *be a binary relation on* $\mathcal{F}$ *that satisfies the axioms of Theorem 10. The following conditions are equivalent:*

(i) $\succsim$ *satisfies Axioms D.1 and D.2;*

(ii) there exist a collection of strictly positive weights $\{\alpha_i\}_{i\in I} \subseteq \Delta(I)$ *and a strictly increasing, continuous function* $v : X \to \mathbb{R}$ *with* $v(0) = 0$ *and* $v(1) = 1$*, such that, for all* $\mathbf{f} \in \mathcal{F}$*,*

$$V_o(\mathbf{f}) = v^{-1}\left(\frac{1}{\alpha_o}\sum_{i\in I}\alpha_i v(c_i(f_i))\right). \tag{8}$$

Moreover, the collection $\{\alpha_i\}_{i\in I}$ *is unique and the function* v *is unique.*

[23]Continuous subjective expected utility and quasi-arithmetic mean models are characterized by Wakker (1988, Theorem 6.2).

By this proposition, *quasi-arithmetic source-dependent preferences* are uniquely represented by triples $((c_i)_{i\in I}, \{\alpha_i\}_{i\in I}, v)$. As before, the certainty equivalents reflect the uncertainty attitude on each source. However, now they are aggregated by means of a weighted sum of utilities. Each weight α_i can be viewed as a measure of the relevance of source i. Note that $\alpha_i > 0$ for all i and, hence, all sources affect the evaluation albeit with a different degree. Furthermore, all weights α_i are normalized with respect to some source o, so the overall evaluation of any prospect profile is computed using the unit of account of that source. Observe that representation (8) is a special case of representation (7) where the rates and the aggregator have the following form:

$$\delta_{ij}(x) = v^{-1}\left(\frac{\alpha_j}{\alpha_i} v(x)\right) \quad \text{and} \quad W_o\left((x_i)_{i\in I}\right) = v^{-1}\left(\sum_{i\in I} v(x_i)\right),$$

for all $x \in X$ and all $(x_i)_{i\in I} \in X^I$. Finally, the form of the utility v captures a preference for smoothing across sources, as shown by our comparative statics analysis later on.

Before moving on, let us recall that the class of quasi-arithmetic means is closely related to classic utility models in economics. Indeed, representation (8) conforms essentially to subjective expected utility if the certainty equivalent functionals are quasi-linear as in Example 3, and the utilities u_i's in that example are such that $u_i = v$ for all $i \in I$.

Standard temporal and allocation settings can be cast as special cases of the quasi-arithmetic source-dependent model. Importantly, we will see that in these applied settings the numeraire earns precise economic content. We next turn to study these special cases.

4.3 Temporal sources

By viewing consumption dates as temporal sources, this section studies intertemporal preferences within our framework of source dependence. A key advantage of taking a source perspective is to allow for time-dependent uncertainty attitudes. The same source of risk (e.g., financial risk) evaluated at different ages or time horizons may affect agents in different ways; this may naturally generate different risk attitudes in a way consistent with the empirical evidence, as argued in the Introduction.

Formally, let $I = \{0, 1, \ldots, T\}$ be interpreted as a set of dates. Then, a prospect profile $\mathbf{f} = (f_0, \ldots, f_T)$ is a temporal stream of consumption where f_t denotes the bundle consumed at time $t \in I$. We impose the following stationarity assumption on inter-source preferences.

Axiom D. 3 *(Stationarity) If $s, t \in I \setminus \{T\}$ and $x, y \in X$, then*

$$(x, s) \sim (y, t) \implies (x, s+1) \sim (y, t+1).$$

The above axiom preserves the well-known behavioral content of stationarity: the indifference between two time-dependent consequences hinges on the dates s and t, whereas it is not affected by postponing consumption of both consequences by the same amount of time. We next show that familiar exponential discounting utility models can be recovered as special cases of our quasi-arithmetic source-dependent representation.

Proposition 13 *Let $\succsim$ be a binary relation on $\mathcal{F}$ that satisfies the axioms of Theorem 10. The following conditions are equivalent:*

(i) $\succsim$ satisfies Axioms D.1, D.2, and D.3;

(ii) there exist a scalar $\beta > 0$ and a strictly increasing, continuous function $v : X \to \mathbb{R}$ with $v(0) = 0$ and $v(1) = 1$, such that, for all $\mathbf{f} \in \mathcal{F}$,

$$V_0(\mathbf{f}) = v^{-1}\left(\sum_{t=0}^{T} \beta^t v(c_t(f_t))\right). \tag{9}$$

Moreover, β and v are unique.

According to representation (9), the agent evaluates a prospect profile $\mathbf{f}$ by first computing the time-t certainty equivalent of each component f_t. Time-t certainty equivalent captures the risk attitude exhibited over time-t horizon. Then, the utility of each time-t certainty equivalent is evaluated from the perspective of time 0 by applying the conversion rate β^t. The overall evaluation is given by the discounted sum of utilities of (time-dependent) certainty equivalents.

From our source perspective, the discount factor is a classic example of subjective price that converts the value of outcomes from a given date to another one. Moreover, the choice of the numeraire o is simple since, as usual in applied temporal contexts, the value of a stream is discounted at time 0. Indeed, representation (9) is a special case of representation (7) where

$$\delta_{ij}(x) = v^{-1}\left(\beta^{j-i} v(x)\right) \quad \text{and} \quad W_0\left((x_i)_{i\in I}\right) = v^{-1}\left(\sum_{i\in I} v(x_i)\right),$$

for all $x \in X$ and all $(x_i)_{i\in I} \in X^I$. The aggregator $W_0(\mathbf{x})$ can be thought of as the value of a perpetuity that is exactly as good as $\mathbf{x}$ itself.

Note that representation (9) reduces to standard discounted utility over constant prospect profiles. Furthermore, one can specify the certainty equivalents as in Example 3 and set $c_t(f_t) = u_t^{-1}\left(\int_S u_t(f_t)\,dp_t\right)$, where u_t is a utility function on X and p_t is a probability measure on S for all $t = 0, ..., T$. In the extreme case in which $u_t = v$ for all $t = 0, ..., T$, we recover the standard discounted expected utility criterion with no source effects and constant risk aversion:

$$V_0(\mathbf{f}) = u^{-1}\left(\sum_{t=0}^{T} \beta^t \int_S u(f_t)\,dp_t\right).$$

We conclude this section by briefly relating our model to some existing works on horizon-dependent risk attitudes. If we consider a set $I = \{0, 1\}$ of only two dates, representation (9) can be rewritten as

$$v^{-1}\left(\left(1-\beta\right)v\left(c_0\left(f_0\right)\right)+\beta v\left(c_1\left(f_1\right)\right)\right),$$

which corresponds essentially to the model studied in Andries et al. (2018) who distinguish between two levels of risk aversion toward immediate and delayed uncertainty, respectively. By allowing for levels of risk aversion that are decreasing with the time horizon, they generate predictions consistent with the empirical evidence on the term structure of equity returns.

5 Comparative attitudes

The additional structure of the quasi-arithmetic model allows us to develop a comparative statics analysis of the effects of inter-source smoothing on choice behavior.

Consider an agent with quasi-arithmetic source-dependent preferences represented by $\left(\left(c_i\right)_{i\in I}, \left\{\alpha_i\right\}_{i\in I}, v\right)$. We say that the agent has a *preference for source smoothing* if, for any $\mathbf{x} = (x_i)_{i\in I} \in X^I$, the agent prefers the profile that yields the quasi-arithmetic v-mean of $\mathbf{x}$ on all sources to $\mathbf{x}$ itself. That is, if we denote by $z = v^{-1}\left(\frac{1}{\alpha_o}\sum_{i\in I}\alpha_i v\left(x_i\right)\right) \in X$ such mean, we must have $(z, z, \ldots, z) \succsim \mathbf{x}$.[24] This definition is reminiscent of the classic notion of risk aversion with an important caveat: our notion of source smoothing applies to prospect profiles that do not involve any risk as they are constant within each source. Risk aversion, instead, is a property of intra-source tastes and is captured by the source-dependent certainty equivalents. The latter do not play any role here and will therefore be omitted in the following analysis.

We now apply the above notion to develop a comparative definition of source smoothing. Consider two agents with respective preference relations given by $\succsim_1$ and $\succsim_2$. We posit that $\succsim_1$ has a *higher preference for source smoothing* than $\succsim_2$ if and only if, for all $\mathbf{x} \in X^I$ and $z \in X$,

$$\mathbf{x} \succsim_1 (z, z, ..., z) \implies \mathbf{x} \succsim_2 (z, z, ..., z).$$

In words, $\succsim_1$ has a higher preference for source smoothing than $\succsim_2$ if, whenever $\succsim_1$ prefers a non-constant outcome profile across sources to a constant one, then the same is true for $\succsim_2$. Next result provides a behavioral characterization and shows that the comparative attitudes toward source smoothing are controlled by the function v.

Proposition 14 *Let $\succsim_1$ and $\succsim_2$ be two quasi-arithmetic source-dependent preferences represented by $\left(\left\{\alpha_i^1\right\}_{i\in I}, v_1\right)$ and $\left(\left\{\alpha_i^2\right\}_{i\in I}, v_2\right)$, respectively. The following conditions are equivalent:*

[24] Here $(z, z, \ldots, z) \in X^I$ is the constant profile that yields outcome z on all sources.

(i) $\succsim_1$ *has a higher preference for source smoothing than* $\succsim_2$*;*

(ii) $\{\alpha_i^1\}_{i\in I} = \{\alpha_i^2\}_{i\in I}$ *and* $v_1 = \psi \circ v_2$ *for some concave, strictly increasing, and continuous transformation* $\psi : v_2(X) \to v_1(X)$*.*

This proposition states that higher preferences for source smoothing are characterized by more concave utility functions v. As anticipated earlier, this result reminds the standard comparative characterization of risk aversion.[25] In turn, our result captures a form of aversion to the variance of outcomes across sources.

When $I = \{0, 1, \ldots, T\}$ is interpreted as a set of dates, the notion of higher preference for source smoothing corresponds to the familiar notion of *higher preference for intertemporal smoothing*. As an immediate corollary of the above proposition, we conclude that in the intertemporal setting of Proposition 13, $\succsim_1$ has a higher preference for intertemporal smoothing than $\succsim_2$ if and only if $\beta_1 = \beta_2$ and v_1 is more concave than v_2.

6 A portfolio allocation problem

This section adopts our quasi-arithmetic model to introduce source dependence into a simple financial allocation problem. We examine the implications on the optimal portfolio allocation and the relationship with some well-known empirical puzzles.

6.1 Market and subjective prices

Consider an agent who faces the following investment problem: there are two assets (f_1, f_2) in two different markets, say Eurozone and US, with prices $(p_1, p_2) \in \mathbb{R}^2_{++}$ and he has to decide how to allocate his wealth $w = (w_1, w_2) \in \mathbb{R}^2_{++}$ across these two assets (prices and wealth are denominated in the local currencies). Each asset is modeled as a random variable $f_i : S_i \to [0, \infty)$ that pays in the local currency and depends on a local source of uncertainty. In view of our theory, we can set $I = \{1, 2\}$, where source 1 (resp., source 2) identifies the Eurozone market (resp., the US market). Each asset f_i corresponds to a source-dependent prospect and $\mathbf{f} = (f_1, f_2)$ is the resulting prospect profile.

The agent can buy $\xi_i \geq 0$ units of asset i (short-sales are not possible), forming in this way a portfolio $\xi = (\xi_1, \xi_2)$. He can move money from one market to the other, once converted in the same currency through a market exchange rate. We denote by ρ_{ij} the market (*nominal*) *exchange rate* from currency j to currency i; that is, ρ_{12} says how many euros the market trades for 1 dollar. In particular, $\rho_{12}\rho_{21} = 1$. The budget constraints are then:

$$\xi_1 p_1 + t \leq w_1 \text{ and } \xi_2 p_2 \leq w_2 + \frac{1}{\rho_{12}} t. \tag{10}$$

[25] See, e.g., Wakker (1989, Definition VII.6.4).

The scalar t is the quantity of money that the agent transfers either from the Eurozone to the US if $t \geq 0$ or vice versa if $t \leq 0$. Here the relevant *real exchange rate* is $e_{12} = \rho_{12} p_2 / p_1$, which states how many units of the Eurozone asset the market trades for 1 unit of the US asset.

Each portfolio ξ determines the random profile $(\xi_1 f_1, \xi_2 f_2)$. The agent evaluates each payoff $\xi_i f_i$ separately, according to

$$c_i(\xi_i f_i) = u_i^{-1}\left(\int_{S_i} u_i(\xi_i f_i)\, dq_i\right), \tag{11}$$

where each u_i is strictly increasing and continuous, and q_i is a probability measure on S_i. The utility functions u_i may be different, so attitudes toward uncertainty are source-dependent. As seen next, we posit that each u_i has a CRRA form so that $c_i(\xi_i f_i) = \xi_i c_i(f_i)$.[26] Using the quasi-arithmetic criterion of Proposition 12, the agent aggregates the evaluations in (11) according to the function $W : \mathbb{R}_+^2 \to \mathbb{R}$ given by

$$W(\xi_1, \xi_2) = v^{-1}\left(\alpha_1 v(\xi_1 c_1(f_1)) + \alpha_2 v(\xi_2 c_2(f_2))\right),$$

where $\alpha_1, \alpha_2 \in (0,1)$ and $\alpha_1 + \alpha_2 = 1$. We assume that the functions u_i's and v have a CRRA form and are strictly concave. Specifically, for each i, there exists $\gamma_i \in (0,1)$ such that

$$u_i(x) = \frac{x^{1-\gamma_i}}{1-\gamma_i} \quad \text{for all } x \geq 0,$$

and there exists $k \in (0,1)$ such that

$$v(x) = \frac{x^{1-k}}{1-k} \quad \text{for all } x \geq 0.$$

Let $\delta_{21} = \delta_{21}(1)$ stand for how many dollars are subjectively equivalent to a payoff of 1 euro for the agent. That is, δ_{ij} is a subjective price and we refer to it as the (nominal) *subjective exchange rate*. By quantifying the subjective value of one currency in terms of the other, δ_{ij} can be interpreted, for instance, as a measure of familiarity toward one market relative to another (which in turn determines how monetary rewards are evaluated in the two markets). Our first result shows that subjective exchange rates play a key role in determining the optimal portfolio allocation.

Proposition 15 *The unique optimal portfolio $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2)$, provided it is an interior solution, is such that*

$$\frac{p_1 \hat{\xi}_1}{\rho_{12} p_2 \hat{\xi}_2} = \left(e_{12} \frac{\delta_{21} c_1(f_1)}{c_2(f_2)}\right)^{\frac{1-k}{k}}, \tag{12}$$

where $\delta_{21} = \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1-k}}$. Moreover, $\frac{\partial W}{\partial \xi_2}(\hat{\xi}) / \frac{\partial W}{\partial \xi_1}(\hat{\xi}) = \rho_{12} p_2 / p_1 = e_{12}$.

[26] To make the problem nontrivial, we also assume that $c_1(f_1), c_2(f_2) > 0$.

The left-hand side of (12) is the ratio of the optimal investments in each market, expressed in euros via the nominal exchange rate. The right-hand side is a function of the product, also expressed in euros, of the objective e_{12} and subjective $\delta_{21} c_1(f_1)/c_2(f_2)$ terms of trade between the two assets. In particular, $\delta_{21} c_1(f_1)/c_2(f_2)$ is the ratio of the subjective values of each euro invested in each market.

The relation between subjective and market exchange rates in (12) reminds the classic relation between subjective discount rates and market interest rates that characterizes Euler equations in intertemporal consumption problems. Indeed, (12) is a spatial counterpart of the classic temporal relation. Across time or space, optimality imposes a tradeoff between market prices and subjective traits.

6.2 Comparative analysis and home bias

The last proposition derives the optimal portfolio allocation of an agent with quasi-arithmetic source-dependent preferences and formalizes the way subjective exchange rates affect financial decisions. In the special case $\delta_{21} = 1$, the agent perceives a payoff of 1 euro as subjectively equivalent to a payoff of 1 dollar and, therefore, he is equally familiar about the value of euros versus dollars. This means that he may still exhibit source-dependent uncertainty attitudes, reflected in the certainty equivalents c_1 and c_2; however, once uncertainty is factored out, he finds monetary outcomes generated by either source as perfectly comparable, analogously to what we have seen in the two-urn Ellsberg experiment.[27] Condition (12) then reduces to

$$\frac{p_1 \hat{\xi}_1}{\rho_{12} p_2 \hat{\xi}_2} = \left(e_{12} \frac{c_1(f_1)}{c_2(f_2)} \right)^{\frac{1-k}{k}}, \tag{13}$$

and the ratio of the optimal investments depends only on the market exchange rate e_{12}. On the other side, if $\delta_{21} > 1$, then a payoff of 1 euro is worth more for the agent than a payoff of 1 dollar. By comparing (12) to (13), we conclude that, *ceteris paribus*, the more δ_{21} exceeds 1, the more the agent will invest in asset 1. As argued in the Introduction, this is a key observation which suggests an explanation of the evidence on home bias and under-diversification in terms of familiarity. Taking the perspective of a Eurozone investor, the higher is the degree of familiarity for the Eurozone in comparison to the US zone (i.e., the higher δ_{21} is), the higher will be the inclination to invest in the Eurozone asset.

The role of δ_{21} is analogous to the one played by a subjective discount factor β in temporal settings. As well known, a discount factor less than 1 indicates that the investor values more the present in comparison to the future. The lower the discount factor is, the higher the (market) real interest rate must be to compensate for the investor's impatience. Here, the higher the subjective exchange rate δ_{21} is, the lower the (real) market exchange

[27] Note that having $\delta_{21} = 1$ does not necessarily imply that the market (nominal) exchange rate ρ_{21} is 1, as well.

rate e_{12} needs to be to compensate for the familiarity bias toward the Eurozone. In a vein akin to the classic temporal Euler equation, these opposing forces can be seen in action in condition (12).

We conclude this section with a comparative statics analysis that studies how each parameter of the problem affects the optimal portfolio allocation. First, the coefficient k is an index of the agent's preference for source smoothing. Indeed, note that the higher k is, the more concave the function v is. By Proposition 14, we can then say that, *ceteris paribus*, the higher k is, the more the agent prefers constant profiles across sources.

Now, denote by

$$\tilde{\xi}_i = \frac{\hat{\xi}_i}{w_1 + \rho_{12} w_2}$$

the optimal share of asset i which measures the fraction of wealth invested in asset i. From expression (12) it can be easily observed that $\tilde{\xi}_i$ depends on the weight α_i on source i, the coefficient of uncertainty aversion γ_i within source i, and the price p_i. Next proposition studies how the optimal share $\tilde{\xi}_i$ of asset i is affected by each of these parameters, keeping everything else equal.

Proposition 16 *Let $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2)$ be the unique optimal portfolio satisfying condition (12) of Proposition 15 and $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)$ be the resulting vector of asset shares. Then,*

$$\frac{\partial \tilde{\xi}_i}{\partial \alpha_i} > 0, \quad \frac{\partial \tilde{\xi}_i}{\partial \gamma_i} < 0, \quad \textit{and} \quad \frac{\partial \tilde{\xi}_i}{\partial p_i} < 0 \quad \textit{for} \quad i = 1, 2.$$

A higher α_i can be interpreted as reflecting a stronger familiarity bias for source i. In turn, a higher weight α_i on source i yields a higher share of asset i. Moreover, a higher coefficient of uncertainty aversion γ_i within source i yields a lower share of asset i. Finally, a higher price p_i for asset i yields a lower share of asset i.

7 Concluding remarks

This paper shows that source preference may originate from two behavioral features represented by source-dependent certainty equivalents (intra-source tastes) and subjective prices (inter-source tastes). The notion of subjective price is a key novelty. It has a precise economic meaning and allows us to characterize preferences over entire prospect profiles. We develop a unifying framework that incorporates existing approaches to study choice under uncertainty in the presence of multiple sources, as discussed in Section 3. We highlight the economic relevance of our theory by showing that classic settings of intertemporal and allocation choices can be viewed as special cases.

We close by noting that our setup does not explicitly account for source-dependent prospects that depend simultaneously on multiple sources. This is compatible with many contexts (notably with intertemporal settings), but it can be restrictive in other circumstances. Though a systematic analysis of this issue is beyond the scope of this paper, our setup can easily be extended to accommodate some forms of multiple dependence. For instance, consider the allocation problem of last section where an agent has to decide how to allocate his wealth between a Eurozone asset f_1 and a US asset f_2. In that application each asset is a source-dependent prospect that depends only on one source, the Eurozone market (source 1) or the US market (source 2). Suppose that an additional source, say the Asian market (source 3), influences the performance of both assets. Indeed, a financial crisis in Asia is likely to affect both the Eurozone and the US. This issue can be addressed within our setup by viewing it as a form of interdependence between sources. Let $\hat{S}_i = S_i \times S_3$ stand for the set of effective contingencies for market $i = 1, 2$. That is, $\hat{S}_i$ describes the relevant uncertainty on market i by taking into account all possible combinations of contingencies featured by the local source i and the Asian source. Asset i is then defined as $\hat{f}_i : \hat{S}_i \to [0, \infty)$ for $i = 1, 2$. As before, $\hat{f}_i$ pays in the local currency, but now it depends on the interactions between the local source (Eurozone or US) and the external source (Asia) that together determine the payoff structure of the asset. Such interdependence directly affects the underlying mechanism of uncertainty and, hence, may alter the source-dependent uncertainty attitudes: the certainty equivalent in (11) is now computed with respect to a probability measure defined on the extended set of contingencies $\hat{S}_i$. The definition of the remaining parameters is formally unchanged. In particular, all exchange rates are still defined between euros and dollars. Any influence of the Asian source on the Eurozone and the US is captured by the respective certainty equivalents and assessed using one of the two currencies. Hence, the analysis of the last section remains valid within this extended setup. A natural direction for future research is the study of more complicated forms of interdependence and their potential impact on the familiarity perception of markets as captured by subjective prices.

Appendix

A Preliminaries on Chisini means

This section introduces the notion of Chisini mean and provides preliminary results relating this notion to our preference representations. The results of this section will be useful to prove the representation theorems in Appendix B.

We start with some terminology. Let Ω be a nonempty set (in this appendix, Ω can be either S or I), Σ an algebra of subsets of Ω (in this appendix, Σ can be either one of the Σ_i's or the power set $\wp(I)$), $F = B_0(\Omega, \Sigma, X)$ the space of all simple Σ-measurable functions $f : \Omega \to X$, (endowed with the supnorm topology $\|\cdot\|$), and $F_c = \{x 1_\Omega : x \in X\}$ the set of constant elements of F.

A functional $V : F \to \mathbb{R}$ is said to:

- be *monotone* if $f \geq g$ implies $V(f) \geq V(g)$;

- be *normalized* if $V(x1_\Omega) = x$ for all $x \in X$;

- *represent* a binary relation $\succsim$ on F if

$$V(f) \geq V(g) \iff f \succsim g.$$

Denote by $\mathcal{V}$ the class of all functionals $V : F \to \mathbb{R}$ with the following two properties:

(χ.1) the restriction of V to F_c is one-to-one;

(χ.2) V is monotone and continuous.

Remark 17 *First note that*

$$x_n 1_\Omega \to x 1_\Omega \text{ in } B_0(\Omega, \Sigma, X) \iff \|x_n 1_\Omega - x 1_\Omega\| \to 0 \iff x_n \to x \text{ in } \mathbb{R}.$$

Therefore property (χ.1) is equivalent to require the function

$$\begin{array}{rccc} v: & X & \to & \mathbb{R} \\ & x & \mapsto & V(x1_\Omega) \end{array}$$

to be one-to-one on X. Together with property (χ.2), this implies that v is strictly increasing and continuous. Moreover, denoting by $\overline{f}$ and $\underline{f}$ the pointwise maximum and minimum of any function $f \in F$ on Ω, monotonicity of V guarantees that $v(\underline{f}) \leq V(f) \leq v(\overline{f})$. Hence, $V(F) = v(X)$.

Definition 18 *Given $V \in \mathcal{V}$, a **Chisini mean** of an element $f \in F$ with respect to V is the (unique) number $M = M_V(f)$ such that*

$$V(M1_\Omega) = V(f).$$

That is, $M \in X$ is the unique solution

$$M_V(f) = v^{-1}(V(f))$$

of the equation $v(M) = V(f)$.

The next simple proposition is a characterization of *Chisini means*:

Proposition 19 *The following conditions are equivalent for a functional $C : F \to X$:*

- *there exists $V \in \mathcal{V}$ such that*

$$C(f) = M_V(f) \qquad \forall f \in F; \tag{14}$$

- *the functional C is monotone, normalized, and continuous.*

Moreover, V is unique up to strictly increasing and continuous transformations.[28]

Proof of Proposition 19. If there exists $V \in \mathcal{V}$ such that $C = M_V$, then

$$f \geq g \implies V(f) \geq V(g) \implies M_V(f) = v^{-1}(V(f)) \geq v^{-1}(V(g)) = M_V(g)$$

and so C is monotone. As to normalization, note that, for all $x \in X$,

$$C(x1_\Omega) = v^{-1}(V(x1_\Omega)) = v^{-1}(v(x)) = x.$$

Moreover, continuity of C follows directly from continuity of V.

Conversely, if C is monotone, normalized, and continuous, then $C \in \mathcal{V}$, that is, it satisfies properties (χ.1) and (χ.2). Define $c : X \to X$ as $c(x) = C(x1_\Omega)$. Then, $c(x) = x$ for all $x \in X$, implying that

$$M_C(f) = c^{-1}(C(f)) = C(f) \qquad \forall f \in F.$$

Hence, C is a Chisini mean with respect to itself.

Finally, assume W is another element of $\mathcal{V}$ such that $C = M_W$. Then, for all $f \in F$,

$$v^{-1}(V(f)) = M_V(f) = M_W(f) = w^{-1}(W(f)) \implies W(f) = \left(w \circ v^{-1}\right)(V(f)).$$

By Remark 17, both $v^{-1} : v(X) = V(F) \to X$ and $w : X \to \mathbb{R}$ are strictly increasing and continuous, so is $\tau = w \circ v^{-1}$. Conversely, let $\tau : V(F) \to \mathbb{R}$ be strictly increasing and continuous, and $W = \tau \circ V$. Then, for all $x \in X$,

$$w(x) = W(x1_\Omega) = \tau(V(x1_\Omega)) = \tau(v(x)) \tag{15}$$

is strictly increasing and continuous because v and τ are. Thus, W satisfies property (χ.1). Monotonicity of W follows immediately from that of V through τ, so that W satisfies property (χ.2). Finally, (15) implies that $\tau = w \circ v^{-1}$. Hence, for all $f \in F$,

$$M_W(f) = w^{-1}(W(f)) = w^{-1}(\tau(V(f))) = w^{-1}\left(\left(w \circ v^{-1}\right)(V(f))\right) = v^{-1}(V(f)) = M_V(f)$$

as desired. ■

Lemma 20 *Let $\succsim$ be a binary relation on F. The following conditions are equivalent:*

(i) there exists a Chisini mean $C : F \to X$ that represents $\succsim$;

(ii) the binary relation $\succsim$ has the following properties:

(π.1) *$\succsim$ is complete and transitive;*

[28] Another element $W \in \mathcal{V}$ is such that $C = M_W$ if and only if there exists a strictly increasing and continuous transformation $\tau : V(F) \to \mathbb{R}$ such that $W = \tau \circ V$. In particular, C is the only normalized element of $W \in \mathcal{V}$ such that $C(f) = M_W(f)$ for all $f \in F$.

(π.2) *if* $f, g \in F$ *and* $f \geq g$ *then* $f \succsim g$. *The converse is true if both* f *and* g *are constant;*

(π.3) *if* $f, g, h \in F$ *and* $f \succ g \succ h$, *then there exist* $\alpha, \beta \in (0,1)$ *such that* $(1-\alpha) h + \alpha g \succ f \succ (1-\beta) h + \beta g$.

Moreover, C *is the unique Chisini mean representing* $\succsim$ *on* F *and it is such that*

$$f \sim C(f) 1_\Omega \quad \forall f \in F. \tag{16}$$

When the elements of F are interpreted as single uncertain payoffs (in this paper, when $F = \mathcal{F}_i$), equation (16) says that the Chisini mean $C(f)$ of f is the single sure payoff which is equivalent to f, that is C is a *certainty equivalent*.
When the elements of F are interpreted as streams of sure payoffs (in this paper, when $F = X^I$), equation (16) says that the Chisini mean $C(f)$ of f is the constant payoff a stream of which is equivalent to f, that is C is an *aggregator*.[29]

Proof of Lemma 20. (i) implies (ii). Let C be a Chisini mean that represents $\succsim$ on F. By Proposition 19, the functional C is monotone, normalized, and continuous. Since X is a convex subset of $\mathbb{R}$, we can apply Proposition 1 of Cerreia-Vioglio et al. (2011), and immediately conclude that $\succsim$ satisfies properties (π.1) and (π.3). Moreover, monotonicity of C implies that

$$h \geq g \implies C(h) \geq C(g) \implies h \succsim g.$$

If $h = x1_\Omega$ and $g = y1_\Omega$ are constant, we have, by normalization of C, that

$$h \succsim g \implies C(x1_\Omega) \geq C(y1_\Omega) \implies x \geq y \implies h \geq g.$$

Hence, property (π.2) holds, too.

(ii) implies (i). We claim that, for all $f \in F$, there exists a unique $x_f \in X$ such that $f \sim x_f 1_\Omega$. Assume $f \sim x1_\Omega$ and $f \sim y1_\Omega$. Then, by transitivity, $x1_\Omega \sim y1_\Omega$, and, by property (π.2), it must be that $x1_\Omega = y1_\Omega$, i.e., $x = y$. This proves uniqueness.

We next prove existence. For all $f \in F$, let $\overline{f}, \underline{f} \in X$ be such that $\overline{f}1_\Omega \geq f \geq \underline{f}1_\Omega$. Then, property ($\pi$.2), again, yields $\overline{f}1_\Omega \succsim f \succsim \underline{f}1_\Omega$. If either $f \sim \overline{f}1_\Omega$ or $f \sim \underline{f}1_\Omega$ existence is guaranteed; otherwise, $\overline{f}1_\Omega \succ f \succ \underline{f}1_\Omega$. The set

$$X_f = \{x \in X : x1_\Omega \succsim f\}$$

contains $\overline{f}$ but not $\underline{f}$ and, if $y \in X_f$ all elements $x \geq y$ also belong to X_f (by property (π.2) and transitivity). Let $x_f = \inf X_f$. If $x_f 1_\Omega \succ f$, then $x_f 1_\Omega \succ f \succ \underline{f}1_\Omega$ and, by property (π.3), there exists $\alpha \in (0,1)$ such that $(1-\alpha) x_f 1_\Omega + \alpha \underline{f} 1_\Omega \succ f \succ \underline{f}1_\Omega$.

[29] Note however that being a Chisini mean is a property of the functional unrelated to whether the functional represents a preference. Instead being a certainty equivalent is indissolubly tied to the existence of a corresponding preference relation.

Hence, $(1-\alpha)\,x_f+\alpha\underline{f}\in X_f$. But, by property ($\pi$.2), we have $x_f>\underline{f}$, which implies $(1-\alpha)\,x_f+\alpha\underline{f}<x_f$, contradicting the fact that x_f is a lower bound of X_f. If $f\succ x_f 1_\Omega$, then $\overline{f}1_\Omega\succ f\succ x_f 1_\Omega$ and, by property (π.3), there exists $\beta\in(0,1)$ such that $\overline{f}1_\Omega\succ f\succ(1-\beta)\,\overline{f}1_\Omega+\beta x_f 1_\Omega$; but, by property ($\pi$.2), $\overline{f}>x_f$, so $(1-\beta)\,\overline{f}+\beta x_f>x_f$. On the other hand, for every $x\in X_f$, $x1_\Omega\succsim f\succ(1-\beta)\,\overline{f}1_\Omega+\beta x_f 1_\Omega$. Thus, transitivity and property (π.2) imply $x>(1-\beta)\,\overline{f}1_\Omega+\beta x_f$, contradicting the fact that x_f is the greatest lower bound of X_f. In conclusion, it must be the case that $f\sim x_f 1_\Omega$.

Next, we show that there exists a Chisini mean that represents $\succsim$. Let $x,y\in X$ be such that $x1_\Omega\succ y1_\Omega$. Then, property (π.2) implies that $x>y$ and, by convexity of X, we have that $(\alpha x+(1-\alpha)z)1_\Omega>(\alpha y+(1-\alpha)z)1_\Omega$ for all $z\in X$ and $\alpha\in(0,1)$. Again by property (π.2), we obtain $(\alpha x+(1-\alpha)z)1_\Omega\succ(\alpha y+(1-\alpha)z)1_\Omega$, and so, $\succsim$ restricted to F_c satisfies Independence. Thus, the axioms of Proposition 1 in Cerreia-Vioglio et al. (2011) are satisfied, implying that there exists a monotone, normalized and continuous functional $C:F\to\mathbb{R}$ that represents $\succsim$ on F. Since C is normalized and monotone, $C(F)=X$. We conclude, by Proposition 19, that the functional $C:F\to X$ is a Chisini mean.

It remains to prove the uniqueness of C. For all $f\in F$ and $x\in X$, normalization of C implies that $f\sim x1_\Omega$ if and only if $C(f)=C(x1_\Omega)=x$. By the above argument, for all $f\in F$, there exists a unique $x_f\in X$ such that $f\sim x_f 1_\Omega$. Hence, $f\sim C(f)1_\Omega$ for all $f\in F$. ■

B Proofs and related analysis

B.1 Proofs of the results of Section 3

Proof of Lemma 2. It follows directly from Lemma 20 by letting $\Omega=S$ and $F=\mathcal{F}_i$ in that lemma. ■

Proof of Proposition 7. (i) implies (ii). Since $\succsim$ satisfies Axiom A.1, it follows from Lemma 2 that there exists a family $(c_i)_{i\in I}$ of certainty equivalents for $(\succsim_i)_{i\in I}$.

Next, fix $i,j\in I$. By Axiom B.1, for every $x\in X$, there exists $y\in X$ such that $(x,i)\sim(y,j)$. Note that the number y is unique. In fact, by transitivity of $\succsim$ on $\mathcal{P}$, if $\hat{y}\in X$ were such that $(x,i)\sim(\hat{y},j)$, then $(\hat{y},j)\sim(y,j)$. That is, $(\hat{y},j)\sim_j(y,j)$, yielding that $y=c_j\,(y)=c_j\,(\hat{y})=\hat{y}$. By the same reasoning, note that if $i=j$, then $y=x$. Thus, we can define a function $\delta_{ji}:X\to X$ as

$$\delta_{ji}\,(x)=y \text{ where } y \text{ is such that } (x,i)\sim(y,j). \tag{17}$$

Note that if $i=j$, then $\delta_{ii}=\mathrm{id}_X$. Since $i,j\in I$ were arbitrarily chosen, we have that $\delta_{ji}:X\to X$, defined as in (17), is a well-defined function. Next, we show that, for all $i,j\in I$, δ_{ji} is a rate and, hence, it satisfies the properties described in Definition 6.

- If $x > z$, then $(x, i) \succ (z, i)$ by Axiom A.1(i) and the definition of $\succsim_i$. By contradiction, assume that $\delta_{ji}(x) \leq \delta_{ji}(z)$. Again, by Axiom A.1(i), it would follow that $(\delta_{ji}(z), j) \succsim (\delta_{ji}(x), j)$. At the same time, by construction, we would have that

$$(z, i) \sim (\delta_{ji}(z), j) \succsim (\delta_{ji}(x), j) \sim (x, i),$$

yielding, by transitivity, that $(z, i) \succsim (x, i)$, a contradiction. This proves that δ_{ji} is strictly increasing.

- By construction, for every $y \in X$, $(y, j) \sim (\delta_{ij}(y), i)$, that is, $y = \delta_{ji}(\delta_{ij}(y))$. Thus, δ_{ji} is onto (and $\delta_{ij} = \delta_{ji}^{-1}$).

- By construction, for every $x \in X$ and $k \in I$, $(x, i) \sim (\delta_{ji}(x), j)$ and $(\delta_{ji}(x), j) \sim (\delta_{kj}(\delta_{ji}(x)), k)$. By transitivity, this implies that $(x, i) \sim (\delta_{kj}(\delta_{ji}(x)), k)$, that is, $\delta_{kj}(\delta_{ji}(x)) = \delta_{ki}(x)$. Thus, $\delta_{ki} = \delta_{kj} \circ \delta_{ji}$.

Now, let $i, j \in I$, $f_i \in \mathcal{F}_i$, and $g_j \in \mathcal{F}_j$. By the previous part of the proof, we have

$$(f_i, i) \sim (c_i(f_i), i), \ (g_j, j) \sim (c_j(g_j), j), \text{ and } (c_j(g_j), j) \sim (\delta_{ij}(c_j(g_j)), i).$$

By transitivity of $\succsim$ on $\mathcal{P}$ and Lemma 2, we obtain that

$$\begin{aligned}(f_i, i) \succsim (g_j, j) &\iff (c_i(f_i), i) \succsim (c_j(g_j), j) \\ &\iff (c_i(f_i), i) \succsim (\delta_{ij}(c_j(g_j)), i) \iff c_i(f_i) \geq \delta_{ij}(c_j(g_j)),\end{aligned}$$

(ii) implies (i). Let $i \in I$ and $f_i, g_i \in \mathcal{F}_i$. Then,

$$(f_i, i) \succsim (g_i, i) \iff c_i(f_i) \geq \delta_{ii}(c_i(g_i)) \iff c_i(f_i) \geq c_i(g_i).$$

Since i was arbitrarily chosen, it follows that $\succsim$ on $\mathcal{P}$ is a complete binary relation. Next, if $(f_i, i) \succsim (g_j, j)$ and $(g_j\ j) \succsim (h_k, k)$, then representation (4) implies that

$$c_i(f_i) \geq \delta_{ij}(c_j(g_j)) \text{ and } c_j(g_j) \geq \delta_{jk}(c_k(h_k)).$$

Since δ_{ij} is increasing and onto and $\delta_{ki} = \delta_{kj} \circ \delta_{ji}$, we have

$$c_i(f_i) \geq \delta_{ij}(c_j(g_j)) \geq \delta_{ij}(\delta_{jk}(c_k(h_k))) = \delta_{ik}(c_k(h_k)),$$

that is, $(f_i, i) \succsim (h_k, k)$, proving that $\succsim$ on $\mathcal{P}$ is transitive, too.

Since c_i is a certainty equivalent functional, we can apply Lemma 2 and directly conclude that $\succsim_i$ on $\mathcal{F}_i$ satisfies Axiom A.1.

Finally, for every $i, j \in I$ and $x \in X$, let $y = \delta_{ji}(x)$. Then,

$$c_j(y) = y = \delta_{ji}(x) = \delta_{ji}(c_i(x)),$$

proving that $(y,j) \sim (x,i)$. Thus, Axiom B.1 is satisfied.

The uniqueness of $(c_i)_{i\in I}$ follows from Lemma 2. Next, assume that $(\hat{\delta}_{ji})_{j,i\in I}$ is another family of functions that, together with $(c_i)_{i\in I}$, represents $\succsim$ on $\mathcal{P}$ as in (4). Fix $i,j \in I$. It follows that, for all $x \in X$,

$$(x,i) \sim (\delta_{ji}(x), j) \text{ and } (x,i) \sim (\hat{\delta}_{ji}(x), j).$$

By transitivity of $\succsim$, we have $(\delta_{ji}(x), j) \sim (\hat{\delta}_{ji}(x), j)$ which, by using the representation, implies $\delta_{ji}(x) = \hat{\delta}_{ji}(x)$. Since i and j were arbitrarily chosen, it follows that $\delta_{ij} = \hat{\delta}_{ij}$ for all $i,j \in I$. ■

Proof of Proposition 8. (i) implies (ii). By Axiom A.2, for every $i \in I$, there are a strictly increasing and continuous function $u_i : X \to \mathbb{R}$ and a probability measure p_i on (S, Σ_i) such that $c_i : \mathcal{F}_i \to X$ represents $\succsim_i$ where $c_i(f_i) = u_i^{-1}\left(\int_S u_i(f_i)\, dp_i\right)$ for all $f_i \in \mathcal{F}_i$. Fix arbitrary $i,j \in I$, $f_i \in \mathcal{F}_i$, and $g_j \in \mathcal{F}_j$ and suppose that $(f_i, i) \succsim (g_j, j)$. By Axiom A.2, we have $(f_i, i) \sim (c_i(f_i), i)$ and $(g_j, j) \sim (c_j(g_j), j)$. By Axiom B.2, $(c_i(f_i), i) \sim (c_i(f_i), j)$. Then, transitivity implies $(c_i(f_i), j) \succsim (c_j(g_j), j)$, that is, $c_i(f_i)1_{S_j} \succsim_j c_j(g_j)1_{S_j}$. Since c_j represents $\succsim_j$, we conclude that $c_i(f_i) \geq c_j(g_j)$.

It is immediate to verify that (ii) implies (i). The uniqueness part follows from standard arguments (see, e.g., Wakker, 1989). ■

B.2 Proofs of the results of Section 4

In the rest of the appendix, we will refer to

$$\mathcal{F}_c = \{\mathbf{f} \in \mathcal{F} : \mathbf{f} = (x_i 1_{S_i})_{i\in I} \text{ for } x_i \in X\}$$

as the set of constant prospect profiles. We endow $\mathcal{F}_c$ with the product topology. Define the map $P : \mathcal{F}_c \to X^I$ by

$$P(\mathbf{f}) = \boldsymbol{\varphi} \quad \text{where} \quad f_i = x_i 1_{S_i} \quad \text{and} \quad \boldsymbol{\varphi}_i = x_i \quad \forall i \in I,$$

and note that P is a well-defined affine homeomorphism.

Then, we can define a binary relation $\succcurlyeq$ on X^I as

$$\boldsymbol{\varphi} \succcurlyeq \boldsymbol{\psi} \iff \mathbf{f} \succsim \mathbf{g}, \tag{18}$$

where $\mathbf{f}, \boldsymbol{g} \in \mathcal{F}_c$, $\boldsymbol{\varphi} = P(\mathbf{f})$, and $\boldsymbol{\psi} = P(\mathbf{g})$.

We will also make use of the following notion for an aggregator.

Definition 21 *Let $D : X^I \to X$ be an **aggregator**.*

*(i) For every $i \in I$, define $D_i : X \to X$ as $D_i(x) = D(x1_i)$ for all $x \in X$. We say that D_i is a **component** of D.*

(ii) We say that $D : X^I \to X$ is a ***commensurable*** *aggregator if D_i is strictly increasing and $D_i(X) = D_j(X)$ for all $i, j \in I$.*

Next, we present an alternative formulation of Theorem 10 in which the aggregator is not normalized with respect to a specific source. We then show that Theorem 10 follows from Theorem 22.

Theorem 22 *Let $\succsim$ be a binary relation on $\mathcal{F}$. The following conditions are equivalent:*

(i) $\succsim$ satisfies Axioms A.1, B.1, and C.1;

(ii) there exist

(a) a family $(c_i)_{i\in I}$ of ***certainty equivalents*** *for $(\succsim_i)_{i\in I}$,*

(b) a family $(\delta_{ij})_{i,j\in I}$ of ***rates****,*

such that, for all $i, j \in I$, $f_i \in \mathcal{F}_i$, and $g_j \in \mathcal{F}_j$,

$$(f_i, i) \succsim (g_j, j) \iff c_i(f_i) \geq \delta_{ij}(c_j(g_j)),$$

(c) a ***commensurable aggregator*** *$D : X^I \to X$, such that, for all $\mathbf{f}, \mathbf{g} \in \mathcal{F}$,*

$$\mathbf{f} \succsim \mathbf{g} \iff D(c_1(f_1), \ldots, c_n(f_n)) \geq D(c_1(g_1), \ldots, c_n(g_n)).$$

In this case, $\delta_{ij} = D_i^{-1} \circ D_j$ for all $i, j \in I$. Moreover, the elements of $(c_i)_{i\in I}$ and $(\delta_{ij})_{i,j\in I}$ are unique, and so is D.

Proof of Theorem 22. (i) implies (ii). Parts (a) and (b) follow from Proposition 7. Define the map $\tilde{T} : \mathcal{F} \to \mathcal{F}_c$ such that $\mathbf{f} \mapsto \tilde{\mathbf{f}}$ where $\tilde{f}_i = c_i(f_i) 1_{S_i}$ for all $i \in I$. Clearly, $\tilde{T}$ is onto. Since $\succsim$ on $\mathcal{F}$ satisfies Axiom C.1(i), we have that, for all $\mathbf{f}, \mathbf{g} \in \mathcal{F}$,

$$\mathbf{f} \succsim \mathbf{g} \iff \tilde{\mathbf{f}} \succsim \tilde{\mathbf{g}}. \tag{19}$$

Define the map $T = P \circ \tilde{T} : \mathcal{F} \to X^I$ and note that it is also onto. Moreover,

$$T(\mathbf{f}) = T(\mathbf{g}) \iff c_i(f_i) = c_i(g_i) \quad \forall i \in I.$$

Since $\succsim$ satisfies Axioms A.1 and C.1, it follows that $T(\mathbf{f}) = T(\mathbf{g})$ implies $\mathbf{f} \sim \mathbf{g}$. Hence, T is a well-defined function.

Using conditions (18) and (19), we obtain that, for all $\mathbf{f}, \mathbf{g} \in \mathcal{F}$,

$$\mathbf{f} \succsim \mathbf{g} \iff \tilde{T}(\mathbf{f}) \succsim \tilde{T}(\mathbf{g}) \iff T(\mathbf{f}) \succcurlyeq T(\mathbf{g}). \tag{20}$$

Next, we show that $\succcurlyeq$ on X^I satisfies properties (π.1)-(π.3) of Lemma 20. Set $F = X^I$ in that lemma. Clearly, $\succcurlyeq$ satisfies property (π.1) as $\succsim$ on $\mathcal{F}$ is a weak order. Let $\mathbf{x}, \mathbf{y} \in X^I$

such that $x_i \geq y_i$ for all $i \in I$. Then, Axiom A.1(i) yields $x_i 1_{S_i} \succsim_i y_i 1_{S_i}$ for all $i \in I$, and Axiom C.1(i) together with (18) imply $\mathbf{x} \succcurlyeq \mathbf{y}$. Thus, the first part of property (π.2) is satisfied. If $\mathbf{x}, \mathbf{y} \in X^I$ are constant and $\mathbf{x} \succcurlyeq \mathbf{y}$, then Axiom C.1(ii) gives $\mathbf{x} \geq \mathbf{y}$. Hence, property ($\pi$.2) holds, too. Property (π.3) follows directly from Axiom C.1(iii).

By Lemma 20, there exists a Chisini mean $D : X^I \to X$ that represents $\succcurlyeq$ on X^I and such that $\mathbf{x} \sim D(\mathbf{x}) 1_I$ for all $\mathbf{x} \in X^I$. Together with condition (20), it implies that

$$\mathbf{f} \succsim \mathbf{g} \iff D(T(\mathbf{f})) \geq D(T(\mathbf{f})) \quad \forall \mathbf{f}, \mathbf{g} \in \mathcal{F}.$$

By Proposition 19, D is monotone, normalized, and continuous. Thus, D is an aggregator.

It remains to show that D is commensurable and that $\delta_{ij} = D_i^{-1} \circ D_j$ for all $i, j \in I$. Let $x, y \in X$ such that $x > y$ and fix an arbitrary $i \in I$. By Axiom A.1(i), we have $x 1_{S_i} \succ_i y 1_{S_i}$. Then, $D(0, \ldots, x, \ldots, 0) > D(0, \ldots, y, \ldots, 0)$, that is, $D_i(x) > D_i(y)$, which proves that D_i is strictly increasing for all $i \in I$.

Let $x \in X$ and $i, j \in I$. Parts (a) and (b) give $(x, j) \sim (\delta_{ij}(x), i)$. Then,

$$D_j(x) = D(0, \ldots, x, \ldots, 0) = D(0, \ldots, \delta_{ij}(x), \ldots, 0) = D_i(\delta_{ij}(x)).$$

We conclude that $\delta_{ij} = D_i^{-1} \circ D_j$. Finally, since δ_{ij} is onto, it directly follows that $D_i(X) = D_j(X) = X$ for all $i, j \in I$. Part (c) is now proven.

(ii) implies (i). By Proposition 7, parts (a) and (b) imply that $\succsim$ satisfies Axioms A.1 and B.1. Since for every $f \in \mathcal{F}$, we can compute $D((c_i(f_i))_{i \in I})$, it follows from part (c) that $\succsim$ on $\mathcal{F}$ is a weak order. Next, if $\mathbf{f}, \mathbf{g} \in \mathcal{F}$ are such that $f_i \succsim_i g_i$ for all $i \in I$, then parts (a) and (b) imply that $c_i(f_i) \geq c_i(g_i)$ for all $i \in I$. Since D is monotone, we can conclude that $D((c_i(f_i))_{i \in I}) \geq D((c_i(g_i))_{i \in I})$, that is, $\mathbf{f} \succsim \mathbf{g}$. Thus, $\succsim$ satisfies Axiom C.1(i). Now, let $\mathbf{x}$ and $\mathbf{y}$ in X^I be constant and such that $\mathbf{x} > \mathbf{y}$. Then, $x = D(\mathbf{x}) > D(\mathbf{y}) = y$, implying that $\mathbf{x} \succ \mathbf{y}$, and Axiom C.1(ii) holds, too. Since D is a Chisini mean, we can apply Lemma 20 with $F = X^I$ and directly conclude that $\succsim$ satisfies Axiom C.1(iii).

Finally, the uniqueness of the elements of $(c_i)_{i \in I}$ and $(\delta_{ij})_{i,j \in I}$ follows from Proposition 7. The functional D is unique by Lemma 20. ■

We now prove Theorem 10 as a corollary of Theorem 22.

Proof of Theorem 10. (i) implies (ii). Parts (a) and (b) follow from Theorem 22. Fix $o \in I$ and define $\tilde{T}_o : \mathcal{F} \to \mathcal{F}_c$ such that $\mathbf{f} \mapsto \tilde{\mathbf{f}}$ where $\tilde{f}_i = \delta_{oi}(c_i(f_i)) 1_{S_i}$ for all $i \in I$. Since each δ_{oi} is onto, it follows that $\tilde{T}_o$ is also onto. Define the map $T_o = P \circ \tilde{T}_o : \mathcal{F} \to X^I$ and note that T_o is onto. Since each δ_{oi} is strictly increasing, we have

$$T_o(\mathbf{f}) = T_o(\mathbf{g}) \iff c_i(f_i) = c_i(g_i) \quad \forall i \in I.$$

Since $\succsim$ satisfies Axioms A.1 and C.1, we have that $T_o(\mathbf{f}) = T_o(\mathbf{g})$ yields $\mathbf{f} \sim \mathbf{g}$. Thus, the map T_o is well defined. By Axiom C.1(i) and condition (18), it follows that $\mathbf{f} \succsim \mathbf{g} \iff T_o(\mathbf{f}) \succcurlyeq T_o(\mathbf{g})$, for all $\mathbf{f}, \mathbf{g} \in \mathcal{F}$.

By Theorem 22, there exists a commensurable aggregator $D : X^I \to X$ such that

$$\mathbf{f} \succsim \mathbf{g} \iff D(T_o(\mathbf{f})) \geq D(T_o(\mathbf{g})) \quad \forall \mathbf{f}, \mathbf{g} \in \mathcal{F}.$$

Since D is commensurable, we have that D_o^{-1} is strictly increasing. Define $W_o : X^I \to X$ as $W_o(\mathbf{x}) = (D_o^{-1}(D(\mathbf{x}))$. Then,

$$\mathbf{f} \succsim \mathbf{g} \iff W_o(T_o(\mathbf{f})) \geq W_o(T_o(\mathbf{g})) \quad \forall \mathbf{f}, \mathbf{g} \in \mathcal{F}.$$

Note that W_o is monotone because D is monotone. Moreover, for any $\mathbf{x} \in X^I$ such that $x_j = 0$ for all $j \neq o$, we have that $W_o(\mathbf{x}) = D_o^{-1}(D(\mathbf{x})) = x_o$ Thus, W_o is o-normalized.

(ii) implies (i). Omitted as it is analogous to the proof of the sufficiency part of Theorem 22. ■

Proof of Proposition 11. (i) Let $((c_i)_{i\in I}, (\delta_{ij})_{i,j\in I}, W_o)$ and $((c_i')_{i\in I}, (\delta_{ij}')_{i,j\in I}, W_o')$ be two source-dependent representations of $\succsim$. By Proposition 7, it follows that $c_i' = c_i$ and $\delta_{ij} = \delta_{ij}'$ for all $i, j \in I$. Moreover, uniqueness of W_o follows directly by construction. Indeed, recall that $W_o = D_o^{-1} \circ D$ and the aggregator D is unique by Theorem 22.

(ii) Let $o \neq \bar{o}$ and consider $((c_i)_{i\in I}, (\delta_{ij})_{i,j\in I}, W_o)$ and $((c_i)_{i\in I}, (\delta_{ij})_{i,j\in I}, W_{\bar{o}})$. By construction, we have that $W_o = D_o^{-1} \circ D$ and $W_{\bar{o}} = D_{\bar{o}}^{-1} \circ D$. Since $\delta_{o\bar{o}} = D_o^{-1} \circ D_{\bar{o}}$ by Theorem 22, we conclude that $W_o = \delta_{o\bar{o}}(W_{\bar{o}})$. ■

Proof of Proposition 12. (i) implies (ii). Note that X^I is a convex set. Moreover, since for each $i \in I$, $x \geq y$ if and only if $x1_{S_i} \succsim_i y1_{S_i}$, it follows that each component is essential (non-null). By Wakker (1988), there exist a collection $\{\alpha_i\}_{i\in I} \subseteq \Delta(I)$ and a continuous function $v : X \to \mathbb{R}$ such that, for all $\mathbf{f}, \mathbf{g} \in \mathcal{F}_c$,

$$\mathbf{f} \succsim \mathbf{g} \iff \boldsymbol{\varphi} \succcurlyeq \boldsymbol{\psi} \iff \sum_{i=1}^{I} \alpha_i v(x_i) \geq \sum_{i=1}^{I} \alpha_i v(y_i),$$

where $f_i = x_i 1_{S_i}$, $g_i = y_i 1_{S_i}$, $\boldsymbol{\varphi}_i = x_i$, and $\boldsymbol{\psi}_i = y_i$ for all $i \in I$. After a positive and affine transformation, we can normalize v by setting $v(0) = 0$ and $v(1) = 1$. Using condition (20), we can conclude that, for arbitrary $\mathbf{f}, \mathbf{g} \in \mathcal{F}$,

$$\mathbf{f} \succsim \mathbf{g} \iff \tilde{T}(\mathbf{f}) \succsim \tilde{T}(\mathbf{g}) \iff T(\mathbf{f}) \succcurlyeq T(\mathbf{g}) \iff \sum_{i=1}^{I} \alpha_i v(c_i(f_i)) \geq \sum_{i=1}^{I} \alpha_i v(c_i(g_i)),$$

where T and $\tilde{T}$ are the maps defined in the Proof of Theorem 22. Let $i \in I$. Observe that

$$x \geq y \iff x1_{S_i} \succsim_i y1_{S_i} \iff x1_{S_i} \succsim y1_{S_i} \iff \alpha_i v(x) \geq \alpha_i v(y).$$

Since $\alpha_i \in [0, 1]$, it follows necessarily that $\alpha_i > 0$. This implies that

$$x \geq y \iff \alpha_i v(x) \geq \alpha_i v(y) \iff v(x) \geq v(y).$$

Since i was arbitrarily chosen, we have that v is strictly increasing and $\{\alpha_i\}_{i\in I} \subseteq \Delta(I)$ is a collection of strictly positive weights. Define $\gamma_j = \frac{1}{\alpha_j} \in (0,\infty)$ for some $j \in I$. It follows that

$$\mathbf{f} \succsim \mathbf{g} \iff \gamma_j \sum_{i=1}^{I} \alpha_i v(c_i(f_i)) \geq \gamma_j \sum_{i=1}^{I} \alpha_i v(c_i(g_i)). \tag{21}$$

Finally, define $\hat{W}_j : X^I \to X$ as

$$\hat{W}_j(\boldsymbol{\varphi}) = v^{-1}\left(\gamma_j \sum_{i=1}^{I} \alpha_i v(\boldsymbol{\varphi}_i)\right) \qquad \forall \boldsymbol{\varphi} \in X^I.$$

Given (21), it is immediate to verify that $\hat{W}_j$ is j-normalized, monotone, and $V_j = \hat{W}_j \circ T$ is a utility function for $\succsim$.

(ii) implies (i). We omit it as it follows from routine arguments.

By applying standard results (see, e.g., Wakker, 1989), the collection $\{\alpha_i\}_{i\in I}$ is unique and the function v is unique up to positive affine transformations. Moreover, the normalization $v(0) = 0$ and $v(1) = 1$ implies that v must be unique. ■

Proof of Proposition 13. (i) implies (ii). By Proposition 7 and since $\succsim$ satisfies Axiom D.3, we have that, for every $i, j \in I \backslash \{T\}$ and every $x, y \in X$,

$$\delta_{ji}(x) = y \iff (x,i) \sim (y,j) \implies (x,i+1) \sim (y,j+1) \implies \delta_{j+1\ i+1}(x) = y = \delta_{ji}(x),$$

that is, $\delta_{j+1\ i+1} = \delta_{ji}$. If we set $j = t$ and $i = t+1$, we have that, for each $t \in I \backslash \{T\}$,

$$\delta_{01}(x) = \delta_{12}(x) = ... = \delta_{t\ t+1}(x) \qquad \forall x \in X.$$

In particular, set $\hat{\delta} = \delta_{01}(1) = \delta_{12}(1) = ... = \delta_{t\ t+1}(1)$. Recall also that for every $i, j \in I$ and for every $x \in X$ the number $\delta_{ji}(x)$ is the unique element in X such that

$$(x,i) \sim (\delta_{ji}(x), j). \tag{22}$$

By Proposition 12, there exist a collection of strictly positive weights $\{\alpha_i\}_{i\in I} \subseteq \Delta(I)$ and a continuous and strictly increasing function $v : X \to \mathbb{R}$ with $v(0) = 0$ and $v(1) = 1$, and $\gamma_0 > 0$ such that

$$V_0(\mathbf{f}) = v^{-1}\left(\gamma_0 \sum_{t=0}^{T} \alpha_t v(c_t(f_t))\right) \qquad \forall \mathbf{f} \in \mathcal{F} \tag{23}$$

is a utility function for $\succsim$. Let $x = 1$. For every $i, j \in I$, we have

$$(\delta_{ji}(1), j) \sim (1, i) \iff \alpha_j v(\delta_{ji}(1)) = \alpha_i v(1) \iff \alpha_j v(\delta_{ji}(1)) = \alpha_i.$$

By setting $i = t+1$ and $j = t$, we have

$$\alpha_{t+1} = \alpha_t v(\delta_{t\ t+1}(1)) \qquad \forall t \in I \backslash \{T\}.$$

Since $\delta_{t\ t+1}(1) = \delta_{01}(1) = \hat{\delta} > 0$ for all $t \in I \backslash \{T\}$, we can define $\beta = v\left(\hat{\delta}\right) > 0$ and obtain[30]

$$\alpha_{t+1} = \alpha_t \beta \qquad \forall t \in I \backslash \{T\}.$$

It follows that $\alpha_{t+1} = \alpha_0 \beta^{t+1}$ for all $t \in I \backslash \{T\}$. By the proof of Proposition 12, we have that $\gamma_0 = \frac{1}{\alpha_0}$. Thus, we can conclude that

$$\begin{aligned} V_0(\mathbf{f}) = v^{-1}\left(\gamma_0 \sum_{t=0}^{T} \alpha_t v\left(c_t\left(f_t\right)\right)\right) &= v^{-1}\left(\sum_{t=0}^{T} \gamma_0 \alpha_t v\left(c_t\left(f_t\right)\right)\right) \\ &= v^{-1}\left(\sum_{t=0}^{T} \gamma_0 \alpha_0 \beta^t v\left(c_t\left(f_t\right)\right)\right) \\ &= v^{-1}\left(\sum_{t=0}^{T} \beta^t v\left(c_t\left(f_t\right)\right)\right), \end{aligned}$$

proving the statement.

(ii) implies (i). We omit it as it is routine.

Finally, β and v are unique by the uniqueness properties of Proposition 12. ■

B.3 Proofs of the results of Sections 5 and 6

Proof of Proposition 14. (i) implies (ii). Using the implication with the same name in Wakker (1989, Theorem VII.6.5), observe that the fact that $\{\alpha_i^j\}_{i \in I} \subseteq \Delta(I)$ for $j = 1, 2$ is a collection of strictly positive weights prevents case (i.b) in that theorem; moreover, the fact that v_2 is strictly increasing prevents case (i.c) there. We conclude that $\{\alpha_i^1\}_{i \in I} = \{\alpha_i^2\}_{i \in I}$ and that there exists a convex, nondecreasing, and continuous $\varphi : v_1(X) \to \mathbb{R}$ such that $v_2 = \varphi \circ v_1$. Note that, for each $x \in X$, $v_2(x) = v_2\left(v_1^{-1}\left(v_1(x)\right)\right)$. Then, for each $t = v_1(x_t) \in v_1(X)$,

$$\varphi(t) = \varphi\left(v_1\left(x_t\right)\right) = v_2\left(x_t\right) = v_2\left(v_1^{-1}\left(v_1\left(x_t\right)\right)\right) = v_2\left(v_1^{-1}(t)\right)$$

and $\varphi = v_2 \circ v_1^{-1}$. As a consequence, $\varphi : v_1(X) \to v_2(X)$ is bijective, strictly increasing, continuous and so is its inverse $\psi = v_1 \circ v_2^{-1} : v_2(X) \to v_1(X)$, which satisfies $v_1 = \psi \circ v_2$. Convexity of φ implies

$$\varphi(\alpha t + (1-\alpha) z) \leq \alpha \varphi(t) + (1-\alpha) \varphi(z) \qquad \forall t, z \in u_1(X), \alpha \in [0,1]$$

$$\Longleftrightarrow$$

$$\alpha t + (1-\alpha) z \leq \varphi^{-1}(\alpha \varphi(t) + (1-\alpha) \varphi(z)) \qquad \forall t, z \in u_1(X), \alpha \in [0,1].$$

[30]Note that $\alpha_{t+1} = \alpha_t v\left(\delta_{t\ t+1}(1)\right) = \alpha_t v\left(\hat{\delta}\right)$ implies

$$v\left(\hat{\delta}\right) = \frac{\alpha_{t+1}}{\alpha_t} > 0.$$

Since $v(0) = 0$ and v is strictly increasing, we have that $\hat{\delta} > 0$.

Now, taking any $x, y \in v_2(X)$, set $t = \psi(x), z = \psi(y) \in v_1(X)$ so that, for each $\alpha \in [0,1]$,

$$\alpha\psi(x) + (1-\alpha)\psi(y) \leq \psi(\alpha\varphi(\psi(x)) + (1-\alpha)\varphi(\psi(y))) = \psi(\alpha x + (1-\alpha) y).$$

Thus, $\psi = v_1 \circ v_2^{-1} : v_2(X) \to \mathbb{R}$ is concave, strictly increasing, continuous, and such that $v_1 = \psi \circ v_2$.

(ii) implies (i). Assume that $\{\alpha_i^1\}_{i\in I} = \{\alpha_i^2\}_{i\in I}$ and $v_1 = \psi \circ v_2$ for some concave transformation ψ. For each $i \in I$, let $\alpha_i := \alpha_i^1 = \alpha_i^2$. Consider $\mathbf{x} \in X^I$ and $z \in X$ such that $\mathbf{x} \succsim_1 (z, z, ..., z)$. By using the representation in Proposition 12, $\sum_{i\in I} \alpha_i v_1(x_i) \geq v_1(z)$. Since $v_1 = \psi \circ v_2$, we have $\sum_{i\in I} \alpha_i \psi(v_2(x_i)) \geq \psi(v_2(z))$. By concavity of ψ, it follows that $\psi\left(\sum_{i\in I} \alpha_i v_2(x_i)\right) \geq \psi(v_2(z))$. Hence, we conclude that $\mathbf{x} \succsim_2 (z, z, ..., z)$. ■

Proof of Proposition 15. For $i = 1, 2$, assume $u_i(x) = x^{1-\gamma_i}/(1-\gamma_i)$ for all $x \geq 0$, where $\gamma_i \in (0,1)$. To maximize W, given the constraints, is equivalent to maximize $\tilde{W} : \mathbb{R}^2_+ \to \mathbb{R}$ defined by $\tilde{W}(\xi_1, \xi_2) = \alpha_1 v(\xi_1 c_1(f_1)) + \alpha_2 v(\xi_2 c_2(f_2))$ subject to (10). Since $\tilde{W}$ is continuous, strictly monotone, and strictly concave,[31] $\tilde{W}$ has a unique maximum $\hat{\xi}$ which satisfies (10) with equalities, which is easily seen to be equivalent to the single constraint

$$\xi_1, \xi_2 \geq 0,\ \xi_1 p_1 + \xi_2 \rho_{12} p_2 = w_1 + \rho_{12} w_2. \tag{24}$$

To ease notation, let $\rho = \rho_{12}$. Set up the Lagrangian

$$\mathcal{L}(\xi_1, \xi_2, s, \lambda_1, \lambda_2, \mu_1, \mu_2) = \tilde{W}(\xi_1, \xi_2) + \lambda_1\xi_1 + \lambda_2\xi_2 + \mu_1(w_1 - \xi_1 p_1 - s) + \mu_2\left(w_2 + \frac{1}{\rho}s - \xi_2 p_2\right),$$

and find $\hat{\xi}$ and $\hat{\lambda}$ and $\hat{\mu}$ that satisfy the following conditions:

(i) $\alpha_i v'\left(\hat{\xi}_i c_i(f_i)\right) c_i(f_i) + \hat{\lambda}_i - \hat{\mu}_i p_i = 0$ for $i = 1, 2$;

(ii) $-\hat{\mu}_1 + \frac{1}{\rho}\hat{\mu}_2 = 0$;

(iii) $\hat{\lambda}_i \geq 0$, $\hat{\mu}_i \geq 0$ and $\hat{\lambda}_i \hat{\xi}_i = 0$ for $i = 1, 2$ as well as $\hat{\xi}_1 p_1 + s = w_1$ and $\hat{\xi}_2 p_2 = w_2 + \frac{1}{\rho}s$.

Then, $\hat{\xi}$ is the maximizer we are looking for. We start our search by assuming $\hat{\lambda}_1 = \hat{\lambda}_2 = 0$ and $\hat{\mu}_1, \hat{\mu}_2 > 0$. Next, since $v(x) = \frac{x^{1-k}}{1-k}$, we have that for $i = 1, 2$,

$$\alpha_i\left(\hat{\xi}_i c_i(f_i)\right)^{-k} c_i(f_i) + \hat{\lambda}_i = \hat{\mu}_i p_i \iff \hat{\xi}_i^k = \frac{\alpha_i c_i(f_i)^{1-k}}{\hat{\mu}_i p_i}$$
$$\iff \hat{\xi}_i = \left(\frac{\alpha_i c_i(f_i)^{1-k}}{\hat{\mu}_i p_i}\right)^{\frac{1}{k}}.$$

Observe also that $\hat{\mu}_2 = \rho\hat{\mu}_1$. We thus have that

$$\hat{\xi}_1 = \left(\frac{\alpha_1 c_1(f_1)^{1-k}}{\hat{\mu}_1 p_1}\right)^{\frac{1}{k}} \quad \text{and} \quad \hat{\xi}_2 = \left(\frac{\alpha_2 c_2(f_2)^{1-k}}{\rho\hat{\mu}_1 p_2}\right)^{\frac{1}{k}}.$$

[31] To avoid technicalities, we also assume that $c_i(f_i) > 0$.

Call $\hat{\mu}_1 = \hat{\mu}$, $c_1(f_1) = c_1$, and $c_2(f_2) = c_2$. We have that

$$\left(\frac{\alpha_1 c_1^{1-k}}{\hat{\mu} p_1}\right)^{\frac{1}{k}} p_1 + s = w_1 \quad \text{and} \quad \left(\frac{\alpha_2 c_2^{1-k}}{\rho \hat{\mu} p_2}\right)^{\frac{1}{k}} p_2 = w_2 + \frac{1}{\rho} s.$$

It follows that

$$\begin{aligned} s = w_1 - \left(\frac{\alpha_1 c_1^{1-k}}{\hat{\mu} p_1}\right)^{\frac{1}{k}} p_1 \quad &\text{and} \quad \left(\frac{\alpha_2 c_2^{1-k}}{\rho \hat{\mu} p_2}\right)^{\frac{1}{k}} p_2 = w_2 + \frac{1}{\rho} s \\ \Longrightarrow \left(\frac{\alpha_2 c_2^{1-k}}{\rho \hat{\mu} p_2}\right)^{\frac{1}{k}} p_2 &= w_2 + \frac{1}{\rho} w_1 - \frac{1}{\rho} \left(\frac{\alpha_1 c_1^{1-k}}{\hat{\mu} p_1}\right)^{\frac{1}{k}} p_1 \\ \Longrightarrow \left(\frac{\alpha_1 c_1^{1-k}}{\hat{\mu} p_1}\right)^{\frac{1}{k}} p_1 + \rho \left(\frac{\alpha_2 c_2^{1-k}}{\rho \hat{\mu} p_2}\right)^{\frac{1}{k}} p_2 &= w_1 + \rho w_2 \\ \Longrightarrow \left(\frac{1}{\hat{\mu}}\right)^{\frac{1}{k}} &= \frac{w_1 + \rho w_2}{\left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 + \rho \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} p_2} \end{aligned}$$

We can conclude that

$$\hat{\xi}_1 = \left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} \frac{w_1 + \rho w_2}{\left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 + \rho \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} p_2} > 0 \tag{25}$$

and

$$\hat{\xi}_2 = \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} \frac{w_1 + \rho w_2}{\left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 + \rho \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} p_2} > 0. \tag{26}$$

This implies that

$$\frac{\hat{\xi}_1}{\hat{\xi}_2} = \frac{\left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}}}{\left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}}} = \left(\frac{\alpha_1 \rho p_2 c_1^{1-k}}{\alpha_2 p_1 c_2^{1-k}}\right)^{\frac{1}{k}} = \left(\frac{\alpha_1}{\alpha_2} \frac{\rho p_2}{p_1}\right)^{\frac{1}{k}} \left(\frac{c_1}{c_2}\right)^{\frac{1-k}{k}}.$$

Thus, we have

$$\frac{\hat{\xi}_1}{\hat{\xi}_2} = \left(\frac{\alpha_1}{\alpha_2} \frac{\rho p_2}{p_1}\right)^{\frac{1}{k}} \left(\frac{c_1}{c_2}\right)^{\frac{1-k}{k}}. \tag{27}$$

Finally, recall that by Proposition 12, $\delta_{21}(x) = v^{-1}\left(\frac{\alpha_1}{\alpha_2} v(x)\right)$ and $v^{-1}(x) = ((1-k)x)^{\frac{1}{1-k}}$. Hence, $\delta_{21} := \delta_{21}(1) = \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1-k}}$. It follows that

$$\frac{\hat{\xi}_1}{\hat{\xi}_2} = \left(\delta_{21} \frac{c_1}{c_2}\right)^{\frac{1-k}{k}} \left(\frac{\rho p_2}{p_1}\right)^{\frac{1}{k}}, \tag{28}$$

proving condition (12).

Now, let us recall that for $i \in \{1,2\}$,

$$\frac{\partial \tilde{W}}{\partial \xi_i}\left(\hat{\xi}\right) = \alpha_i c_i \left(\hat{\xi}_i c_i\right)^{-k}.$$

Hence,

$$\frac{\frac{\partial \tilde{W}}{\partial \xi_2}\left(\hat{\xi}\right)}{\frac{\partial \tilde{W}}{\partial \xi_1}\left(\hat{\xi}\right)} = \frac{\alpha_2 c_2 \left(\hat{\xi}_2 c_2\right)^{-k}}{\alpha_1 c_1 \left(\hat{\xi}_1 c_1\right)^{-k}} = \frac{\alpha_2}{\alpha_1}\left(\frac{c_2}{c_1}\right)^{1-k}\left(\frac{\hat{\xi}_2}{\hat{\xi}_1}\right)^{-k}.$$

Using (28) and rearranging, we obtain that

$$\frac{\frac{\partial \tilde{W}}{\partial \xi_2}\left(\hat{\xi}\right)}{\frac{\partial \tilde{W}}{\partial \xi_1}\left(\hat{\xi}\right)} = \frac{\rho p_2}{p_1}.$$

Since $W = g \circ \tilde{W}$ where g is such that $g(x) = [(1-k)x]^{\frac{1}{1-k}}$ for all $x \in \mathbb{R}$, it follows that

$$\frac{\partial W}{\partial \xi_i}\left(\hat{\xi}\right) = g'\left(\tilde{W}\left(\hat{\xi}\right)\right)\frac{\partial \tilde{W}}{\partial \xi_i}\left(\hat{\xi}\right) \qquad \forall i \in \{1,2\}.$$

Note that $\tilde{W}\left(\hat{\xi}\right) > 0$, thus $g'\left(\tilde{W}\left(\hat{\xi}\right)\right) > 0$. We can conclude that

$$\frac{\frac{\partial W}{\partial \xi_2}\left(\hat{\xi}\right)}{\frac{\partial W}{\partial \xi_1}\left(\hat{\xi}\right)} = \frac{g'\left(\tilde{W}\left(\hat{\xi}\right)\right)\frac{\partial \tilde{W}}{\partial \xi_2}\left(\hat{\xi}\right)}{g'\left(\tilde{W}\left(\hat{\xi}\right)\right)\frac{\partial \tilde{W}}{\partial \xi_1}\left(\hat{\xi}\right)} = \frac{\rho p_2}{p_1},$$

proving the second part of the proposition. ■

Proof of Proposition 16. The optimal portfolio allocation $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2)$ is given by expressions (25) and (26) in the proof of Proposition 15. Then, the optimal portfolio share $\tilde{\xi}_i = \frac{\hat{\xi}_i}{w_1 + \rho_{12} w_2}$ for each asset i is given by

$$\tilde{\xi}_1 = \frac{\left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}}}{\left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 + \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} \rho p_2} \quad \text{and} \quad \tilde{\xi}_2 = \frac{\left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}}}{\left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 + \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} \rho p_2}$$

To ease notation, set $d = \left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 + \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} \rho p_2$ and note that $d > 0$. Then

$$\begin{aligned}
\frac{\partial \tilde{\xi}_1}{\partial \alpha_1} &= \frac{1}{d^2}\left[\frac{1}{k}\alpha_1^{\frac{1-k}{k}}\left(\frac{c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} d - \frac{1}{k}\alpha_1^{\frac{1-k}{k}}\left(\frac{c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 \left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}}\right] \\
&= \frac{1}{d^2}\left[\frac{1}{k}\left(\frac{\alpha_1^{1-k} c_1^{1-k}}{p_1}\right)^{\frac{1}{k}}\left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} \rho p_2\right] > 0.
\end{aligned}$$

Since $\frac{\partial c_1}{\partial \gamma_1} < 0$, we have that

$$\frac{\partial \tilde{\xi}_1}{\partial \gamma_1} = \frac{1}{d^2}\left[\frac{k}{1-k} c_1^{\frac{2k-1}{1-k}} \left(\frac{\alpha_1 k}{p_1}\right)^{\frac{1}{1-k}} \left(\frac{k\alpha_2 c_2^k}{\rho p_2}\right)^{\frac{1}{1-k}} \rho p_2\right] \frac{\partial c_1}{\partial \gamma_1} < 0.$$

Analogously, it can be shown that $\frac{\partial \tilde{\xi}_2}{\partial \alpha_2} > 0$ and $\frac{\partial \tilde{\xi}_2}{\partial \gamma_2} < 0$.

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