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Just Enough or All: Selling a Firm

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Abstract

We consider the problem of selling a firm to a single buyer. The magnitude of the post-sale cash flow rights as well as the benefits of control are the buyer's private information. In contrast to research that assumes the private information of the buyer is one-dimensional, the optimal mechanism is a menu of tuples of cash-equity mixtures. We provide sufficient conditions such that the optimal mechanism attains one of the following forms: *i)* a take-it or leave-it offer for the smallest fraction of the company that facilitates the transfer of control, or *ii)* a take-it or leave-it offer for *all* the shares of the company. The first case prevails when the seller wants to screen more precisely with respect to the private benefits, while the latter case prevails when the seller wants to screen finely with respect to cash flows.

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1   Introduction and Related Literature

A fundamental question in economics is how the terms of trade are reached in exchange of assets. Although there has been a substantial amount of research that relates the form of negotiations to agreement outcomes for assets with consumption value, much less is known about the optimal mixture of equity and cash when selling corporate control. A private equity company may want to buy the control rights of a firm, with the intention of restructuring the organization to enhance its value and profit from its resale. Typically the private equity's ability to increase the value of the firm is highly uncertain from the perspective of the outsiders. Moreover, the benefits of the takeover to the private equity are partly determined by its financing and operational costs, which are also not known to the outsiders. How then should the owners of the target firm sell their firm? Should they claim rights to the post-takeover profits by retaining some shares of the company? Or should they sell the whole firm? The same question arises, for instance, when a revenue-maximizing government must privatize a company, when a successful startup that has grown large sells itself, and when a large blockholder sells his stake. It is the ability to retain a share of the profits that would be generated by ceding control to a buyer that distinguishes the sale of a firm from many other assets.

Assuming the value of the firm under new ownership is the buyer's private information, theory predicts that the share retained should be the minimal necessary to transfer control of the firm to the buyer (see, for example, Cremer (1987), Samuelson (1987) and Cornelli and Li (1997)). The intuition is that retaining a share in the firm allows the seller to capture the full value of the firm on the retained share and limit the information rents of the buyer to the share that was sold. This intuition may break down, as we show, when the buyer enjoys a private benefit of control or when the buyer has an opportunity cost of controlling the company.

In our setting a single agent—the seller—owns the firm and he can sell any fraction of his shares. The seller is faced with a single buyer (she). The buyer adds value (v) to the target firm if she gains a controlling stake of the firm. In addition she enjoys a private benefit from controlling the firm (b). When the private benefits of control are negative, they represent the opportunity cost of the buyer from controlling the company, and when v is negative, the buyer destroys value. Both v and b are known only to her. Each share has one vote, and for convenience we assume that the controlling stake is 50% of the shares. As a result, the buyer controls the firm only when she obtains a controlling stake of the firm.[1] We identify the optimal mechanism the seller should employ to sell the firm when he can offer the buyer a menu of tuples of fractions of shares and a price per share.

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[1]The analysis remains the same except for some minor changes for any threshold different than 50%.

In our model, neither private benefits b nor cash flows v are explicitly contractible. The tools that the seller has to his disposal are the fraction of shares he transfers and a transfer price, both of which depend solely on the buyer's report and not on the actual value of v or b.[2] A similar assumption restricting contractability of cash flows is implicitly made throughout the related literature as for example in Zingales (1995), Cornelli and Li (1997), and DeMarzo et al. (2005). Unlike in DeMarzo et al. (2005), securities such as debt contracts are superfluous in our model, because there is no moral hazard and neither of the participants is liquidity or budget constrained.

The presence of both types of private information gives rise to an optimal mechanism, which need not take the form of a take-it-or-leave-it offer. In general it will be a non-trivial menu of cash and equity mixtures. This is in contrast to existing research, which assumes the private information of the acquirer is one-dimensional. In that case the optimal selling mechanism is a take-it-or-leave-it offer for half of the company.[3] We provide sufficient conditions on the joint distribution of v and b such that the optimal mechanism takes one of the following forms:

1. A take-it-or-leave-it offer for the smallest fraction of the company that facilitates the transfer of control (Proposition 3).

2. A take-it-or-leave-it offer for *all* the shares of the company (Proposition 4).

We also exhibit an example where the optimal mechanism is a take-it-or-leave-it offer for a fixed fraction of the company in excess of that required to facilitate the transfer of control, but less than 1. Our results suggest that if negotiation for the control of a company settles on a transfer of more than the minimum controlling stake of the firm, we should expect that the buyer enjoys private benefits of control or has opportunity costs of controlling the firm.

We also show that under some conditions, the seller is able to extract the full value, v, per share (Proposition 5), whereas the buyer earns information rents only on the private

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[2]Cash flows v are implicitly contractible in the sense that the owner can retain some shares. Notice, however, that our contracts are not necessarily linear because the amount of shares sold can vary with reported v and b. For complete contractibility on the agent's ex-post payoffs, see Mezzetti (2004) and Deb and Mishra (2013).

[3]Our discussion revolves around the case where the seller is selling the company to a single buyer. When restricted to a single buyer, the results in Cornelli and Li (1997) boil down to selling the smallest share of the company needed for control. However, they do have results showing that under some conditions when the private value is a linear function of the post-takeover value of the company and there are several buyers the seller might be better off by offering a more complicated mechanism involving sale of more than the minimal necessary share for control. Even under those conditions, in contrast to our model, it is never optimal to sell the whole company.

benefits of control. This presents an interesting contrast to Zingales (1995) where it is argued that the two components of firm value (v and b) should be sold using two separate mechanisms. The cash flow rights should be auctioned off to dispersed shareholders. To quote Zingales:

> "The market for cash flow rights, populated by a large number of small investors, is fully competitive. As a result, the incumbent, by selling cash flow rights to dispersed shareholders is able to fetch their full value under the buyer."

He further argues that the private benefits of control, however, should be bargained over in a direct negotiation. A key takeaway from our result is that it is not always necessary to rely on a competitive market of outside shareholders to extract the cash flow rights from a buyer. In some instances the seller can do that with an appropriate mechanism.

In the optimal mechanism, the seller may sell a fraction of the company that is strictly greater than the minimum controlling stake. This is in contrast to models in which either only the common value component is private information, or the private benefits are perfectly correlated with the common value component. The intuition is the following: had the seller known the common value component, v, then the optimal allocation rule would have excluded buyers with low private benefits from the control. There are, however, many mechanisms that would implement the optimal allocation, including those where the seller sells a large stake of the company. When the common value component is the buyer's private information, then the seller may use the *leeway* he has in the choice of the fraction of the company he offers, which is costless to him, to induce the buyer to reveal the common value component for free.

Our analysis commences by showing that every incentive-compatible mechanism induces a corresponding *exclusion boundary* separating the types who obtain a controlling stake from those who do not. The shape of the exclusion boundary is closely related to the incentive compatibility constraints. In particular, the exclusion boundary of an incentive-compatible mechanism is a non-increasing, and in the interior of the valuation space, concave function that maps the post-takeover values of the company, v, into the private valuations, b. Its derivative is equal to the negative of the fraction of the firm the buyer obtains when taking over the control, and is therefore bounded between -1 and -0.5. This restricts the seller's ability to screen. For example, he is unable to sell the firm only to types with the private benefits above some threshold value simply because he is unable to distinguish between a type with a low cash flow v and a high private

benefit b and a type with a high cash flow v and a low b. The security that screens over b most finely is the one in which the seller transfers the minimum amount of shares that is needed for the control of the company. We also show that any boundary satisfying certain properties is the exclusion boundary of some incentive-compatible mechanisms (see Proposition 1).[4]

We further show that the seller's payoff from any incentive-compatible mechanism can be written as a function of the allocation rule along the exclusion boundary and the lowest type's payoff. More precisely, the seller's payoff is the expected value of the virtual valuations above the exclusion boundary, which can be interpreted as the value of screening over the private values, minus an additional term, which corresponds to the information rent the seller has to pay for the agent to truthfully reveal the post-takeover value of the firm. This establishes a form of revenue equivalence for our environment.

Replicating the usual approach adopted in a single-dimensional mechanism design, one would solve the unconstrained problem by giving a controlling stake of the company only to the types with nonnegative virtual valuations. However, two problems arise from trying to apply such an approach directly. The boundary that separates the types whose virtual valuations are nonnegative from the types whose virtual valuations are negative might fail to be a boundary that stems from an incentive-compatible mechanism.[5] An additional challenge is the term corresponding to the information rents paid due to screening over the post-takeover values. Our main results are then obtained using comparisons of exclusion boundaries arising from incentive-compatible mechanisms with the boundaries separating positive virtual valuations from the negative virtual valuations, while minding the additional term for information rents.

Our analysis resembles the analysis undertaken in Lewis and Sappington (1988) and Laffont et al. (1987) in the sense that we reduce a multi-dimensional mechanism design problem to a single-dimensional object, namely an exclusion boundary.[6] However, incentive constraints put restrictions on the shape of the exclusion boundaries, which is particular to our environment. Strong results about the structure of optimal

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[4]Characterization of boundaries arising from and defining incentive-compatible mechanisms is strictly speaking not a full characterization of incentive compatibility, as opposed to the one in Rochet (1987), because there is not a unique incentive-compatible mechanism that induces a given exclusion boundary. Implementability conditions have also been provided in more specialized environments, for example, by McAfee and McMillan (1988). For a recent take on implementability in multi-dimensional environments, see Carbajal and Ely (2010), Kos and Messner (Forthcoming) and Rahman (2010).

[5]We impose restrictions on the distribution of types such that virtual valuation is non-decreasing in the private value for each post-takeover value of the firm. Therefore, the boundary separating the types with non-negative virtual valuations from the ones with negative virtual valuations is a non-increasing function that maps the post-takeover values into private values.

[6]For a more indepth outline of the literature, see Rochet and Stole (2003) and the literature review in Manelli and Vincent (2007).

mechanisms, as the ones obtained in this paper, have remained rather elusive in general multi-dimensional environments. Some notable exceptions are Laffont et al. (1987), Lewis and Sappington (1988), Rochet (2009) and the results in the literature on budget constraints as in Che and Gale (2000) and Pai and Vohra (2008).

The key feature of our model is that both the post-takeover value of the firm and the private benefits of control are the buyer's private information. Beginning with Berle and Means (1932), there has been a substantial literature arguing that control of the firm allows the controller to enjoy benefits not shared with minority shareholders (see Jensen and Meckling (1976), Dodd and Warner (1983) and Johnson et al. (2000)). These benefits can be monetary, such as excess salary, or non-monetary, such as amenities like professional sports teams and newspapers.[7] For example, Dyck and Zingales (2004) estimate that on average the private benefit of control is worth 14% of the equity value of a firm.[8,9]

Given the magnitude of the private benefits of control, there has been a great deal of interest in understanding its impact on the sale of corporate control (see, for example, Bebchuk (1994) and Zingales (1995)), and our contribution is to that literature. We focus on firms whose shares are concentrated in the hands of a controlling shareholder. Many publicly traded small and mid-sized companies in the US and across the world have concentrated ownership (Betton et al. (2008), Holderness (2009)). The sale of firms in which no shareholder has a controlling interest is interesting and has been studied in the context of public tender offers.[10]

Our assumption of a single potential buyer in negotiations is realistic and quite commonly observed. For example, Boone and Mulherin (2007) found in a sample of 400 large merger transactions from the 1990s, involving a total of over one trillion dollars in transaction value, that half were executed through bilateral negotiations. Importantly, the assumption allows us to examine the effect on the selling price and quantity of shares

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[7]Private benefits can be negative because of personal monitoring costs or lawsuits brought by government officials. Another way to think of a negative private benefit is that it can be the outside option of the acquirer or foregone opportunities due to managing the target firm.

[8]See Barak and Lauterbach (2011) for a brief summary of the empirical literature devoted to estimating the magnitude of the private benefit.

[9]There are plenty of other reasons why an acquiring company may have private benefits of control that are not appropriated by the target company's shareholders. For example, the target company may have good distribution capabilities in new areas, which the acquiring company can use for its own products as well. Alternatively, the target company may allow the acquiring company to enter a new market without having to take on the risk, time and expense of starting a new division. Finally, a takeover can facilitate the acquiring company to reduce its redundant functions.

[10]This raises free-riding issues that can emanate from the transfer of control. See, for example, Grossman and Hart (1980), Marquez and Yılmaz (2008), Bagnoli and Lipman (1988), Burkart and Lee (2010) and Marquez and Yılmaz (2012). Ekmekci and Kos (2012) study takeovers when a minority stake of the firm is owned by large shareholders and a majority stake of the firm is widely dispersed.

traded when the buyer's private information is, as in this case, two-dimensional. The literature that allows for more than one buyer assumes that potential buyers either care only about v or assumes that their private benefit of control is perfectly correlated with v.

Our analysis of the optimal mechanism highlights a difficulty in one of the empirical approaches used to estimate private benefits of takeovers (see Barclay and Holderness (1989)). It estimates the private benefits of control via the difference between the share price of a publicly traded firm on the day of the transfer and the following day. Let P be the price paid for control, Q the number of shares transferred and w the value of the shares the following day. Then, the private benefit of control, b, is assumed to be $P - wQ$. This assumption is valid if we believe the seller has the ability to make a take-it-or-leave-it offer and there is no private information. Dyck and Zingales (2004) suggest a modification assuming that bargaining power is shared between buyer and seller but continue to assume full information. Suppose, instead, we maintain the assumption that the seller has all the bargaining power but the buyer has private information about b and the increase in the share value, v. Then, the approach taken in Barclay and Holderness (1989) ignores the informational rent that goes to the buyer. If we account for the informational rent the buyer earns, our model predicts that if v and b are private information but independent of each other, the expected value of b conditional on control being acquired declines with v.

An important feature of optimal mechanisms in our model is that the larger is the v, the larger is the fraction of shares sold. In other words, when the seller retains a smaller fraction of equity, then post-transfer share prices increase more. This is consistent with the empirical studies that document that in mergers and takeovers, if the cash transfer is higher in the cash-equity mixture, the target share price increases more (see Travlos (1987) and Franks et al. (1988)). In our model, this result is obtained due to the two-dimensional private information that the buyer holds. Other models explain this empirical finding either via competition effects on the buyers' side or by two-sided asymmetric information (Hansen (1987), Fishman (1989), Berkovitch and Narayanan (1990), Eckbo et al. (1990)).

Below we introduce the notation we use as well as a description of the model. The remaining sections contain the analysis of the model. All proofs not contained in the main text are in the Appendix.

2 The Model

The value of the company under the current owner is commonly known and normalized to 0. Namely, his plans and management skills are well understood. If, however, the buyer obtains at least a 0.5 share in the company, she obtains control rights. We call this event a successful takeover. The value of the company after a successful takeover (the post-takeover value) is the buyer's private information and resides in an interval $V = [\underline{v}, \overline{v}]$. We allow for $\underline{v} < 0$ in which case the acquisition is value destroying. In addition to the post-takeover value of the company, conditional on a takeover, the buyer obtains a private benefit $b \in [\underline{b}, \overline{b}] = B$, which is also the buyer's private information. We allow for $\underline{b} < 0$. Negative b can be interpreted to mean that control is costly or that the buyer has an unknown outside option that pays off $-b$.

The values of v and b are drawn from some commonly known distribution on $[\underline{v}, \overline{v}] \times [\underline{b}, \overline{b}]$. It will be convenient to think of nature as first choosing the post-takeover value v from a distribution F. We assume that F has density f positive everywhere on $[\underline{v}, \overline{v}]$. Conditional on v, nature chooses b according to distribution $G(b; v)$. The latter has a density $g(b; v)$, which is strictly positive on all of $[\underline{b}, \overline{b}]$ for all v.

If the buyer obtains a share $x \geq 0.5$, she controls the company. In this case the value of the company to her is vx. In addition she enjoys a private benefit b, independently of the shares owned, and pays a transfer t. If she obtains a share $x < 0.5$, that share is worth 0, which is the value of the company under the current owner; in that case she does not enjoy the private benefit. Formally, if the current owner sells to the buyer a share x at a price t, the buyer's utility is

$$u(x, v, b) - t = \begin{cases} vx + b - t & \text{if} \quad x \geq 0.5 \\ -t & \text{if} \quad x < 0.5 \end{cases}.$$

If the owner sells less than 50% of the firm, he receives the transfer t and retains control of the company, the value of which remains 0. If, however, he sells at least a share $x \geq 0.5$, the value of the remaining share is $(1-x)v$. Formally, the seller's payoff is

$$w(x, v) + t = \begin{cases} (1-x)v + t & \text{if} \quad x \geq 0.5 \\ t & \text{if} \quad x < 0.5 \end{cases}.$$

Prior work has examined three special case of the setting we have just described.

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1. v is common knowledge and b is private information.

   If v is common knowledge, the seller can set a price of v per share and capture the entire post-takeover value of the company. That is, he can sell a fraction $x \geq 0.5$ for xv and in addition enjoy $(1 - x)v$ on the share he keeps. Thus, the seller is indifferent about what fraction of the firm beyond 50% is to be given up. The only question is how much of the private benefit of control can be captured. The answer follows from Mussa and Rosen (1978) or Myerson (1981). The optimal mechanism can be seen as a price of v per share and a take-it or leave-it offer for control of the firm. Notice that the model is silent about the fraction of shares beyond 50% that would change hands.

2. b is common knowledge and v is private information.

   In this case the seller will charge b for control and must decide what fraction of shares above 50% to part with and the price to charge for them. An answer was provided by Cremer (1987) and Samuelson (1987) in an environment introduced by Hansen (1985) and generalized in Cornelli and Li (1997). The seller should part with the smallest fraction of shares necessary to yield control for a posted price. This is akin to the observation that it is better to run an auction with contingent payments, i.e., shares. See DeMarzo et al. (2005), Che and Kim (2010) and Skrzypacz (2012) for a more extensive discussion. Limiting the fraction of shares that goes to the buyer limits the informational rents she earns.

3. v and b are private information but linearly related.

   This case was studied in Cornelli and Li (1997) and Cornelli and Felli (2012) . Their results state that with one buyer, the seller makes a take-it-or-leave-it offer for 50% of the shares. The intuition for this result is very similar to the case where b is common knowledge and v is private information.

3 Analysis

We examine the question of how the seller should design an optimal mechanism to sell his firm. Invoking the revelation principle, we can restrict attention to direct mechanisms

(Q, T), where

$$Q : V \times B \to [0, 1],$$

and

$$T : V \times B \to \mathbb{R}.$$

Q is an allocation rule mapping each announced type (v, b) into a fraction of shares of the firm, $Q(v, b)$, the buyer receives. T are the transfers to the seller.[11]

Let

$$P(v, b) = 1_{[Q(v,b) \geq 0.5]}$$

for all (v, b), where 1 is the indicator function. $P(v, b) = 1$ if the buyer upon reporting (v, b) acquires a controlling share of the company and 0 otherwise.

Let (Q, T) be a mechanism. Type (v, b)'s payoff when she reports type (v', b') is

$$P(v', b') \left[vQ(v', b') + b\right] - T(v', b').$$

Incentive compatibility can then be expressed as

$$P(v, b) \left[vQ(v, b) + b\right] - T(v, b) \geq P(v', b') \left[vQ(v', b') + b\right] - T(v', b') \qquad (1)$$

for all (v, b) and (v', b'), and individual rationality as

$$P(v, b) \left[vQ(v, b) + b\right] - T(v, b) \geq 0 \qquad (2)$$

for all (v, b). Let $U(v, b) \equiv P(v, b) \left[vQ(v, b) + b\right] - T(v, b)$. The seller's problem is to maximize

$$E[T(v, b) + vP(v, b)(1 - Q(v, b))],$$

where the expectation is with respect to v and b, over all the mechanisms (Q, T) satisfying (1) and (2).

3.1   Characterization of Incentive Compatible Mechanisms

In a single-dimensional mechanism design problem, under standard single-crossing conditions, an allocation rule Q is implementable if and only if it is non-decreasing. In our setting with two-dimensional types, incentive compatibility implies that Q is non-decreasing in each component (holding the other fixed). This weak form of monotonicity

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[11]Transfers can be assumed deterministic without loss of generality. The assumption that the allocation rule is deterministic is standard in multi-dimensional mechanism design literature, but not without loss.
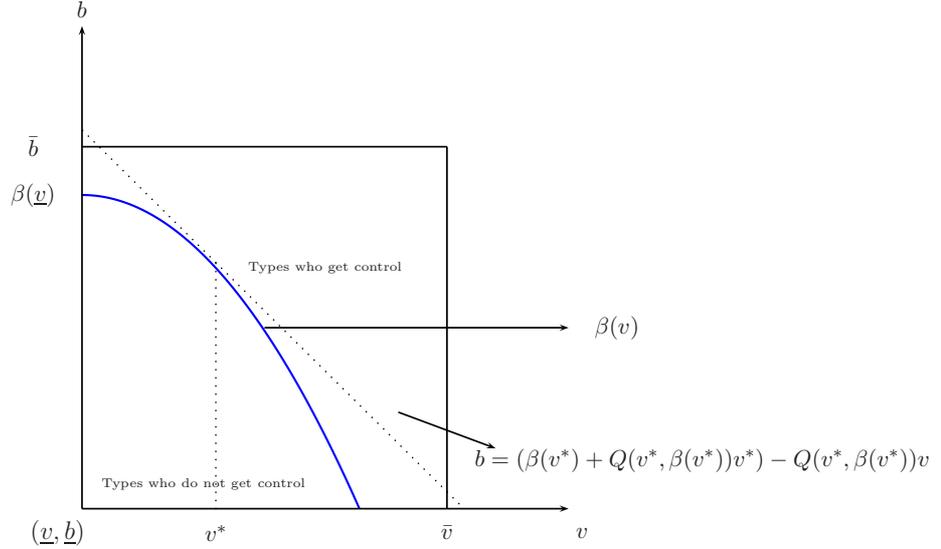
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Figure 1: The curve $\beta(v)$ is the exclusion boundary that separates the types who get control of the firm and the types who get excluded from taking control of the firm. The slope of $\beta(v)$ at $v = v^*$ is equal to $-Q(v^*, \beta(v^*))$.

is insufficient to characterize all incentive-compatible allocation rules. It does allow us, however, to establish existence of a boundary that separates the set of types who obtain control from the set of the types who do not.

Lemma 1. *Let (Q,T) be an incentive-compatible mechanism. Define*

$$\beta(v) = \inf\{b : Q(v,b) \geq 0.5\}$$

for all v. Then $P(v,b) = 1$ if $b > \beta(v)$, and $P(v,b) = 0$ if $b < \beta(v)$, for all (v,b).[12] Moreover, β is nonincreasing.

For each incentive-compatible mechanism (Q,T), we call the associated function $\beta(\cdot)$ the *exclusion boundary* of the mechanism. We say that an exclusion boundary β is *implementable* if there is an incentive-compatible mechanism (Q,T) whose exclusion boundary is β.

Definition 1. *An exclusion boundary β is regular if there are numbers $a, c \in [\underline{v}, \bar{v}]$ such that $\beta(v) = \bar{b}$ for $v < a$; $\beta(v) = \underline{b}$ for $v > c$; β is continuous everywhere and is concave in $[\underline{v}, c]$; and, at every differentiability point $v \in [a, c]$,*

$$\beta'(v) \in [-1, -0.5].$$

The following proposition gives a characterization of all incentive-compatible mechanisms through their exclusion boundaries.

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[12]We adopt convention $\beta(v) = \bar{b}$ if $Q(v,b) < 0.5$ for all b.

Proposition 1. *An exclusion boundary β is implementable if and only if it is regular. If (Q, T) is incentive compatible with an exclusion boundary β, and if β is differentiable at some $v \in [a, c]$ for which $Q(v, \beta(v)) \geq 0.5$, then*

$$\beta'(v) = -Q(v, \beta(v)).^{13}$$

Proposition 1 states that if a mechanism (Q, T) is incentive compatible, its exclusion boundary is regular. Conversely, any regular exclusion boundary is implementable. We show the necessity part of the proposition by first establishing an intermediate result (Lemma 3 in the Appendix). This intermediate result shows that if (Q, T) is an incentive-compatible mechanism and (v, b) is a type such that $Q(v, b) \geq 0.5$, then all the types (v', b') who do not receive a controlling stake must lie below the line with a slope $-Q(v, b)$ that passes through the point (v, b). This is the line that represents the set of types whose payoffs would be identical to the payoff of type (v, b) if they mimicked (v, b). Therefore, any type above this line who does not get a controlling stake in the company would prefer to represent herself as the type (v, b). This reasoning, applied to the exclusion boundary β, yields the second claim and the necessity result stated in Proposition 1 (see Figure 1 for a depiction).

Sufficiency is obtained by constructing a mechanism that allocates the fraction $-\beta'(v^-)$ to types (v, b) such that $v \leq c$,[14] and fraction $-\beta'(c^-)$ when $v > c$. The transfers are, $T(v, b) = 0$ for $v < a$, or $v \in [a, c]$ and $b < \beta(v)$. $T(v, b) = vQ(v, \beta(v)) + \beta(v)$ for $v \in [a, c]$ and $b \geq \beta(v)$, and $T(v, b) = cQ(c, \underline{b}) + \underline{b}$ for $v > c$.

The above result relates the shape of the exclusion boundary β arising from an incentive-compatible mechanism and the allocation rule. In particular, an exclusion boundary β arising from an incentive-compatible mechanism allows us to partition $[\underline{v}, \bar{v}]$ into three intervals. If $a > \underline{v}$, then $\beta(v) = \bar{b}$ when $v \leq a$. On the second interval, which we call the interior segment, $\beta(v)$ is strictly decreasing with a derivative equal to $-Q(v, \beta(v))$ almost everywhere on this interval. If $c < \bar{v}$, then $\beta(v) = \underline{b}$ for all $v \in [c, \bar{v}]$. Given an incentive-compatible mechanism, one or more of the segments may be empty. Moreover, standard monotonicity considerations imply that $Q(v, \beta(v))$ is nondecreasing in v. Therefore, β is concave on the interior segment.

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[13]Strictly speaking, if v is a differentiability point of β and if $v \in [a, c]$, then $\beta'(v) = -\tilde{Q}$ for any $\tilde{Q}$ that is a limit point of $Q(v, b')$ as $b' \downarrow b$. We keep the current formulation that is stated when $Q(v, \beta(v)) \geq 0.5$ for expositional simplicity.

[14]$\beta'(v^-)$ denotes the left-hand derivative of β at v.

3.2 The Seller's Profit from an Incentive Compatible Mechanism

Now we turn to the seller's problem, which is to maximize his profits among all incentive-compatible mechanisms. We begin with a preliminary result that will be of use later. First we characterize the buyer's payoff when the seller screens her only along the dimension of private benefits.

Lemma 2. *Let (Q, T) be an incentive-compatible mechanism. Then,*

$$U(v, b) = U(v, \underline{b}) + \int_{\underline{b}}^{b} P(v, x) dx.$$

Proof. After observing that

$$U(v, b) \geq U(v, b') + [b - b'] P(v, b'),$$

the proof follows the approach of Myerson (1981). $\qquad\square$

Lemma 2 shows that along the dimension of private benefits, the indicator that the buyer obtains a controlling stake plays the same role as the probability of obtaining the object in a standard auction setting. Moreover, one should notice that this relation is independent of the fine details of the allocation Q, beyond the P.

The next result gives an expression for the seller's profit arising from an incentive-compatible mechanism (Q, T), solely in the terms of the allocation rule Q and the payoff to the lowest type, thereby establishing a revenue-equivalence result.[15]

Proposition 2. *Let (Q, T) be an IC mechanism. Then the seller's payoff from mechanism (Q, T) can be written as*

$$\pi = -U(\underline{v}, \underline{b}) + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db - P(v, \underline{b}) Q(v, \underline{b}) \frac{1 - F(v)}{f(v)} \right\} f(v) dv. \tag{3}$$

The expression for the seller's profit can be decomposed into two parts. The term $v + b - \frac{1 - G(b; v)}{g(b; v)}$ has the standard interpretation of a conditional virtual value. It represents the amount the seller is able to capture from the type (v, b) conditional on the post-takeover value being v. The seller is unable to capture all of the buyer's private benefits

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[15]The analysis follows standard arguments. See, for example, the derivation in Armstrong (1996).

b as he needs to incentivize the buyer to truthfully reveal the component b by leaving her some information rent.

The seller's revenue is further diminished by ignorance of the post-takeover value of the firm v. The additional information rent he has to pay for the revelation of v is captured in the term $-P(v, \underline{b})Q(v, \underline{b})\frac{1-F(v)}{f(v)}$.

The seller's payoff is, at least directly, dependent on Q only along the ray $b = \underline{b}$.[16] However, we show below that the controlling stake in the firm that the buyer receives, Q, does play a significant role in the characterization of incentive-compatible mechanisms. Namely, the shape of Q puts restrictions, through incentive compatibility, on the shape of P, which enters the payoff directly.

3.3   Structure of Optimal Mechanisms

Let the conditional virtual valuation of type (v, b) be

$$\phi(v, b) \equiv v + b - \frac{1 - G(b; v)}{g(b; v)}.$$

As mentioned earlier, $\phi(v, b)$ is the virtual valuation of type (v, b) if the seller knew the common value component, v. We make the following single-crossing assumption: $\phi(v, \cdot)$ crosses 0 from below at most once for every $v \in V$.

Let

$$\alpha(v) \equiv \inf\{b : \phi(v, b) \geq 0\}$$

for all $v \in V$. The term $\alpha(v)$ is the smallest b, given a fixed v, for which the virtual valuation $\phi(v, b)$ is nonnegative. If, for a given v, $\phi(v, b) < 0$, for all b we set $\alpha(v) = \underline{b}$ (see Figure 2 for a depiction).

Assumption 1. α is absolutely continuous and piecewise differentiable, and $\alpha'(v) \geq -0.5$ at each point of differentiability v.

Assumption 1 is a restriction on the distribution of types, $G(b; v)$. Below we give examples of where Assumption 1 is satisfied. Our first main result provides a characterization of optimal mechanisms under Assumption 1.

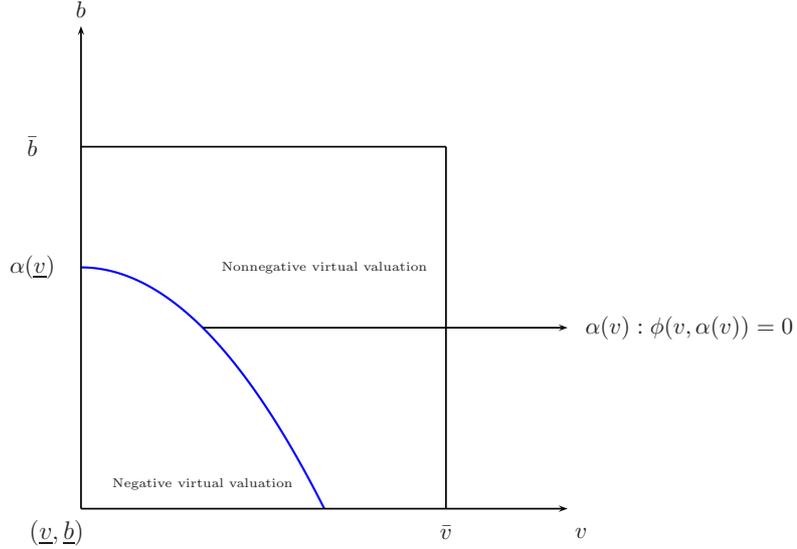[16]Indirectly, Q makes an appearance in P.

Figure 2: The curve $\alpha(v)$ separates the types whose conditional virtual valuations are nonnegative from those with virtual valuations that are negative.

Proposition 3. *If Assumption 1 is satisfied, a take-it-or-leave-it offer for half of the firm is an optimal mechanism for the seller.*

When Assumption 1 is satisfied the seller would like to screen more finely with respect to b than with respect to v. The intuition is most clear in the extreme case where α is flat ($\alpha' = 0$). In such a case the seller would like to screen solely on the basis of b. He would like to sell only to the agents with b exceeding some threshold. The seller can, however, not distinguish between a type that has a high v and low b and a type that has a low v and a high b. Indeed, our characterization of incentive compatibility implies that the 'flattest' mechanism the seller can offer, and thus the finest screening over b, is achieved by a take it or leave it offer for a half of the company.

In what follows we provide further details on how the result in Proposition 3 is obtained. It is convenient to use the following definitions:

$$\pi_0(v) \equiv \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db,$$

for all v, and

$$\pi(v) = \pi_0(v) - P(v, \underline{b}) Q(v, \underline{b}) \frac{1 - F(v)}{f(v)}. \tag{4}$$

Then, according to Proposition 2,

$$\pi = -U(\underline{b}, \underline{v}) + \int_{\underline{v}}^{\bar{v}} \pi(v) f(v) dv.$$

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The term $\pi_0(v)$ is the standard representation of the seller's revenue in a one dimensional setting. In our setup it has an interpretation as the revenue the seller would obtain if he knew the v component of the buyer's type and had to screen only the private benefit component, b. There is an additional component of the seller's payoff in the multi-dimensional setting, namely $-P(v,\underline{b})Q(v,\underline{b})\frac{1-F(v)}{f(v)}$. This component is present due to screening along the v dimension. In a single-dimensional setting, an analogous term is taken care of by setting the lowest type's payoff to some constant. In our case, a single type's payoff can also be set to a constant, $(\underline{v},\underline{b})$, which leaves a ray $\{(v,\underline{b}) : v \in (\underline{v},\overline{v}]\}$ whose payoffs are linked to the type $(\underline{v},\underline{b})$. The additional cost for screening those types is precisely $P(v,\underline{b})Q(v,\underline{b})\frac{1-F(v)}{f(v)}$. Notice that this cost needs to be paid only if the type $(v,\underline{b})$ receives a controlling stake of the company, as his payoff is otherwise zero.

The idea of the proof of Proposition 3 can now be described as follows. Function α from V to B maximizes $\pi_0(v)$ pointwise. The proof shows that for each incentive-compatible mechanism $(\tilde{Q},\tilde{T})$ and its exclusion boundary $\tilde{\beta}$, there exists another boundary β^* corresponding to a take-it-or-leave-it offer for half of the company such that $\beta^*(v)$ is between $\alpha(v)$ and $\tilde{\beta}(v)$ for every v. The single-crossing assumption implies that $\pi_0^*(v) \geq \tilde{\pi}_0(v)$ for all v. Indeed, if, for example, $\alpha(\hat{v}) \leq \beta^*(\hat{v}) \leq \tilde{\beta}(\hat{v})$ for some $\hat{v}$, then neither $\tilde{\beta}$ nor β^* assign a controlling stake to types with the post-takeover value $\hat{v}$ and a negative virtual valuation. However, $\tilde{\beta}$ might exclude some types with a positive virtual valuation that β^* does not. Moreover, by construction β^* has a smaller or equal cost of screening along the ray $\{(v,\underline{b}) : v \in (\underline{v},\overline{v}]\}$ than does $\tilde{\beta}$. Therefore, $\pi^*(v) \geq \tilde{\pi}(v)$. The existence of such a β^* is possible due to the assumption that α has a slope greater than or equal to -0.5 everywhere in the interior of $V \times B$, while $\tilde{\beta}$ has a slope in $[-1, -0.5]$, from the incentive compatibility of the mechanism.

Below, we give a simple sufficient condition on distributions under which Assumption 1 is satisfied and therefore Proposition 3 applies.

Corollary 1. *Suppose v and b are distributed independently, with distributions F and G, respectively. Suppose G has a continuously differentiable density g on $[\underline{b},\overline{b}]$, such that $g(b) > 0$ and $g'(b) \geq 0$ for all $b \in [\underline{b},\overline{b}]$. Then a take-it-or-leave-it offer for half of the firm is an optimal mechanism for the seller.*

Proof. Since $g'(b) \geq 0$ for all b,

$$\phi_b(v,b) = 2 + \frac{g'(b)}{g^2(b)}(1 - G(b)) > 0.$$

This implies that $\phi(v,\cdot)$ crosses 0 at most once and from below.

Define a function $\hat{\alpha} : [\underline{v}, \bar{v}] \to \mathbb{R}$ by

$$v + \hat{\alpha}(v) - \frac{1 - G(\hat{\alpha}(v))}{g(\hat{\alpha}(v))} = 0$$

for every $v \in [\underline{v}, \bar{v}]$. Now define a function α by truncating $\hat{\alpha}$ to $[\underline{b}, \bar{b}]$.[17]

For any v such that $\alpha(v) \in \{\underline{b}, \bar{b}\}$ and α is differentiable, $\frac{d\alpha}{dv}(v) = 0$. For any v such that $\alpha(v) \in (\underline{b}, \bar{b})$, differentiating

$$v + \alpha(v) - \frac{1 - G(\alpha(v))}{g(\alpha(v))} = 0$$

with respect to v yields

$$\frac{d\alpha(v)}{dv} = -\frac{g(\alpha(v))}{2g(\alpha(v)) + (v + \alpha(v))g'(\alpha(v))} \geq -\frac{1}{2}.$$

Since α, g and g' are continuous, the derivative $\frac{d\alpha}{dv}$ is also bounded from above, rendering α Lipschitz continuous. Thereby Assumption 1 is satisfied and Proposition 3 applies. $\square$

In the following Proposition, we show that if the zero virtual valuation curve is very steep, and if it does not hit the lower boundary, then a take-it-or-leave-it offer for the whole company is optimal for the seller.

Proposition 4. *Let α be absolutely continuous and piecewise differentiable with $\alpha'(v) \leq -1$ at each point of differentiability $(v, \alpha(v))$ in the interior of $V \times B$, and $\alpha(\bar{v}) > \underline{b}$. Then a take-it-or-leave-it offer for the whole firm is an optimal mechanism for the seller.*

The intuition for Proposition 4 is as follows: Ideally, the seller would use a mechanism whose exclusion boundary coincides with the zero virtual valuation curve, α. However, the slope of the exclusion boundary of an incentive-compatible mechanism is bounded below by -1. Therefore, the seller will benefit by using a mechanism whose exclusion boundary has slope -1, i.e., a take-it-or-leave-it offer for the whole firm. In other words, this mechanism screens most finely across cash flows v. The assumption that $\alpha(\bar{v}) > \underline{b}$ guarantees that the optimal take-it or leave-it offer for the whole firm will exclude all types with a private benefit $\underline{b}$.

In our model some caution is required in trying to adapt the approach used in the single-dimensional case to our setting. In a single-dimensional environment, under single

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[17]If $\hat{\alpha}(v) > \bar{b}$, then $\alpha(v) = \bar{b}$. If $\hat{\alpha}(v) < \underline{b}$, then $\alpha(v) = \underline{b}$.

crossing, one solves a relaxed problem and then shows that the mechanism that serves only types with nonnegative virtual valuations can be implemented in an incentive-compatible and individually rational mechanism. One could try to do the same in our case. Function α serves that purpose. However, two difficulties arise. First, α can be of a form that cannot arise as an exclusion boundary of an incentive-compatible mechanism. For example, if α is linear with a slope $-1/4$, no IC mechanism exists whose boundary is α, due to Proposition 1. Second, even if α is such that it could arise as an exclusion boundary in an IC mechanism, it might not solve the seller's problem. The hiccup is that α maximizes π_0 point by point rather than π. However, if α takes the form that could arise from an IC mechanism and $\alpha(\bar{v}) > \underline{b}$, then the corresponding mechanism has $P(v, \underline{b}) = 0$ for all v. In this case the solution to the relaxed problem indeed yields a solution of the original problem. We demonstrate this in the next proposition and in the example that follows it.

Assumption 2. α is regular and $\alpha(\bar{v}) > \underline{b}$.

Proposition 5. *If Assumption 2 is satisfied, then the optimal mechanism's exclusion boundary is $\alpha(v)$. Moreover, the seller's expected profits in this mechanism are identical to his expected profits if he employs the optimal mechanism after observing v.*

Proposition 5 asserts that the seller's expected profits are equal to the maximum expected profits had he known the common-value component. It highlights that the ability of the seller to claim a fraction of the post-transaction profits gives him the ability to perfectly screen the common-value component of the buyer's information. In such a mechanism, the buyer retains informational rents only on his private benefits.

In light of Propositions 1 and 2, Proposition 5 follows straightforwardly. If the zero virtual valuation curve is regular and does not hit the lower boundary, then excluding only the types who have negative virtual valuations yields an implementable exclusion boundary. Clearly the seller cannot do better than this. Inspection of the seller's profit function given by equation (3) shows that setting $P(v, \underline{b}) = 0$ and setting $P(v, b) = 1$ only when $\phi(v, b) \geq 0$ yields the maximum profit for the seller ignoring the incentive compatibility conditions implied by the exclusion boundary.

4   Examples

In this section, we explicitlly solve for the optimal mechanism when the private benefits (b) are distributed according to a uniform distribution and when they are distributed

according to an exponential distribution.

Example 1. *(b is uniformly distributed and is independent of v)* Let b and v be independent, where b is distributed uniformly on $[0, \bar{b}]$ and v with some atomless distribution F on $[0, 1]$. It is easy to verify that

$$\alpha(v) = \frac{1}{2} - \frac{1}{2}v.$$

A take-it-or-leave-it offer for half of the company at the price $1/2$ generates an exclusion boundary equal to α. Moreover, this mechanism has $P(v, 0) = 0$ for all v except $\bar{v}$. Therefore, $\pi(v)$ is maximized for all v except $\bar{v}$. As the distribution over v is atomless we verify the optimality of a take-it-or-leave-it offer for half of the company at the price $1/2$.

Example 2. Let b and v be independent, where b is distributed according to an exponential distribution on $[0, \infty)$ with a hazard rate λ, i.e., $G(b; v) = 1 - e^{-\frac{b}{\lambda}}$. The cash flow v is distributed according to some atomless distribution F on $[0, 1/\lambda]$. It is easy to verify that

$$\alpha(v) = \frac{1}{\lambda} - v.$$

A take-it-or-leave-it offer for the whole company at the price $1/\lambda$ generates an exclusion boundary equal to α.

The above examples illustrate the optimality of two simple mechanisms, a take-it or leave-it offer for half or for the whole company. In general, the optimal mechanism need not take any of these two forms, as shown in the example below. In fact, it need not even be a take-it-or-leave-it offer.

Example 3. Let v be distributed uniformly on $[0, 2/3]$. The details of the distribution of v are not important. Nature first chooses v and then b from a uniform distribution on $[0, 1 - v/2]$. Therefore,

$$g(b; v) = \frac{1}{1 - v/2},$$

and

$$G(b; v) = b\frac{1}{1 - v/2}.$$

The adjusted virtual valuation is then easily computed to be

$$\phi(v, b) = \frac{3}{2}v + 2b - 1.$$

And finally,

$$\alpha(v) = \frac{1}{2} - \frac{3}{4}v.$$

The optimal mechanism can now be implemented as a take-it-or-leave-it offer for 3/4 of the company at the price of 1/2.

5 Conclusion

The market for corporate control is different from many other exchange markets in which consumers trade with producers. Shares of a company carry both cash-flow rights and voting rights. Therefore, transferring control requires a minimum transfer of 50% of the shares. The seller may sell all the shares or alternatively claim a minority portion of the profits that result from the transfer. In this paper, we analyze the optimal mechanism for the sale of such a firm to a buyer whose private information is two-dimensional. We find that, unlike in the case where the uncertainty is unidimensional, the optimal mechanism may not be a take-it-or-leave-it offer. In general it is a menu of tuples of fractions of shares and cash transfers. In particular, we identify sufficient conditions on the joint distribution of v and b for which the optimal mechanism takes one of the following forms:

1. A take-it-or-leave-it offer for the smallest fraction of the company that facilitates the transfer of control.

2. A take-it-or-leave-it offer for *all* the shares of the company.

We also identify a sufficient condition for the seller to extract the full value, v, per share so that the buyer earns information rents only on the private benefits of control. The main insight is that the seller can reduce the information rents the buyer enjoys on the benefits of control through the quantity of shares retained.

Although our model is a stylized model, our analysis also applies to the case of a large block holder with control rights of a firm who is negotiating with a potential buyer for the sale of the control rights. If the fraction of shares the block holder needs to sell for the transfer of control is less than his total holdings, then a slight modification of our analysis applies. Our results imply that, in private negotiations for block trades, the block holder can use the fraction of shares he transfers to better screen the buyer's type.

A Proofs

Proof of Lemma 1. Let (Q, T) be incentive compatible. First we show a preliminary result, which will be of use later in the proof.

Claim: If $Q(v, b) \geq 0.5$ for some (v, b), then $Q(v', b') \geq 0.5$ for all $(v', b') \geq (v, b)$. Proof of the Claim: Let (v, b) be such that $Q(v, b) \geq 0.5$; if no such (v, b) exists, we are done. Let $(v', b') \geq (v, b)$ and $(v', b') \neq (v, b)$. Incentive compatibility implies

$$u(Q(v, b), v, b) - u(Q(v', b'), v, b) \geq u(Q(v, b), v', b') - u(Q(v', b'), v', b');$$

equivalently,

$$u(Q(v, b), v, b) - u(Q(v, b), v', b') \geq u(Q(v', b'), v, b) - u(Q(v', b'), v', b').$$

Since $Q(v, b) \geq 0.5$ and consequently $P(v, b) = 1$,

$$Q(v, b)[v - v'] + b - b' \geq u(Q(v', b'), v, b) - u(Q(v', b'), v', b').$$

If $Q(v', b') < 0.5$, the right-hand side equals 0, while the left-hand side is negative, resulting in a contradiction. Thus, $P(v', b') = 1$ and $Q(v', b') \geq 0.5$, thereby concluding the proof of the claim.

Returning to the original Lemma, let $v \in [\underline{v}, \bar{v}]$ and $\beta(v)$ be defined as in the statement of the Lemma. The claim above implies that $P(v, b') = 1$ for every $b' > \beta(v)$, and $P(v, b') = 0$ for every $b' < \beta(v)$.

Suppose $v > v'$ and $\beta(v) > \beta(v')$. Then there exists a $\hat{b} \in (\beta(v'), \beta(v))$ such that $Q(v, \hat{b}) < 0.5$ and $Q(v', \hat{b}) \geq 0.5$, which contradicts the above Claim. Therefore, β is nonincreasing. $\square$

Lemma 3. *Let (Q, T) be an incentive-compatible mechanism, and let (v, b) be such that $Q(v, b) \geq 0.5$. If $P(v', b') = 0$, then*

$$-Q(v, b)v' + (Q(v, b)v + b) \geq b'.$$

Proof. As above, incentive compatibility implies

$$u(Q(v, b), v, b) - u(Q(v, b), v', b') \geq u(Q(v', b'), v, b) - u(Q(v', b'), v', b'),$$

which then yields

$$Q(v, b)[v - v'] + b - b' \geq 0,$$

or equivalently

$$-Q(v, b)v' + (Q(v, b)v + b) \geq b'.$$

$\square$

Lemma 3 implies the following. Let $(v, \beta(v)) \in V \times B$. Then if (v', b') is such that $P(v', b') = 0$, it has to be that (v', b') is below the linear function with the slope $-Q(v, \beta(v))$, which runs through $(v, \beta(v))$.

Proof of Proposition 1. *If β is implementable, then it is regular.*

Let (Q, T) be an incentive-compatible mechanism whose exclusion boundary is β, and let S be the set of types (v, b) for which $Q(v, b) \geq 0.5$. For any type $t = (v, b) \in S$, let $L(t) := \{(v', b') : Q(v, b)v' + b' < Q(v, b)v + b\}$. Consider the set $L := \cap_{t \in S} L(t)$. For any type $(v, b) \in L$, $P(v, b) < 0.5$ because the sets L and S are disjoint. For any set X, let $cl(X)$ denote the closure of X. We will show that for any $(v, b) \notin cl(L)$, $P(v, b) = 1$. Clearly, if $L = \emptyset$, or if $cl(L) = V \times B$, then the claim is true. So, we suppose that $L \neq \emptyset$ and $cl(L) \neq V \times B$.

Notice that L is a convex set because it is an intersection of convex sets. Clearly, its closure is also convex. Pick a type $t' = (v', b') \notin cl(L)$. Then, we claim there exists a type (v, b) such that $Q(v, b)v' + b' > Q(v, b)v + b$. By way of a contradiction, suppose not. Since L is convex, for any type $t'' = (v'', b'') \in L$, there is a $\lambda \in (0, 1)$ such that the type $\lambda t' + (1 - \lambda)t'' \notin cl(L)$. But this is a contradiction because for any $(v, b) \in S$, $Q(v, b)(\lambda v' + (1 - \lambda)v'') + \lambda b' + (1 - \lambda)b'' < Q(v, b)v + b$.

As there is a type $(v, b) \in S$ such that $Q(v, b)v' + b' > Q(v, b)v + b$, and since $P(v, b) = 1$, by Lemma 3 (in the Appendix), $P(v', b') = 1$. Therefore, if S is the set of types who get control of the firm, then $cl(L) \cup S = V \times B$, and L and S are disjoint.

Since the exclusion boundary of β separates the types who get control and those who do not, β has to be the boundary of L. Since L is convex, its boundary β is concave in the interval in which $\beta(v) > \underline{b}$. We have already shown that β is nonincreasing; therefore, there is a number $c \in [\underline{v}, \bar{v}]$ such that β is concave on $[\underline{v}, c)$ and $\beta(v) = \underline{b}$ for $v > c$. Because β is concave on $[\underline{v}, c)$, it is continuous, and almost everywhere differentiable on this interval. To complete the proof of continuity on the whole domain, we need to show that if $c < \bar{v}$, then $\lim_{v \to c} \beta(v) = \underline{b}$. Assume on the way to a contradiction that this is not true. Then $c < \bar{v}$ and there is an $\epsilon > 0$ such that $\beta(v) > \underline{b} + \epsilon$ for every $v < c$ because β is nonincreasing. But then the type $(c - \epsilon/2, \underline{b} + \epsilon)$ would prefer to mimic a type (v', b') that satisfies $v' \in (c, c + \epsilon/2)$ and $b' \in (\underline{b}, \underline{b} + \varepsilon)$ for some $\varepsilon < c + \epsilon/2 - v'$, which yields the desired contradiction.

Since β is continuous and nonincreasing, it is differentiable almost everywhere. Let $v \in (a, c)$ be a point of differentiability of β. Pick a $v' < v$. Since $(v', \beta(v')) \in cl(L)$, $Q(v, b)v + b(v) \geq Q(v, b)v' + \beta(v')$ for every $b > \beta(v)$. Because $Q(v, b) \in [0.5, 1]$ for every $b > \beta(v)$, it has to be that $\beta(v') - \beta(v) \leq v - v'$. Since this is true for every $v' < v$, it has to

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be that $\beta'(v) \geq -1$. Similarly, by picking $v' > v$, we argue that $\beta(v) - \beta(v') \geq 0.5(v' - v)$, establishing that $\beta'(v) \geq -0.5$.

We have thus proven that β is regular.

If β is regular, then it is implementable.

Let β be a regular boundary. Let (a, c) be the largest interval in which $\beta(v) > \underline{b}$. Because β is concave on $v \in [\underline{v}, c]$, it is differentiable almost everywhere on the same interval. This, together with the assumption that β is a regular boundary, implies $\beta'(v) \in [-1, -0.5]$ almost everywhere on $v \in [\underline{v}, c]$. Let $\beta'(v^-)$ denote the left-hand derivative of β at v. Because β is concave in $[\underline{v}, c]$, $\beta'(v^-)$ exists at every point $v \in (\underline{v}, c)$ and is nonincreasing in v. Note that $\beta'(v^-) = \beta'(v)$ whenever $\beta'(v)$ exists. Because $\beta'(v) \in [-1, -0.5]$ almost everywhere in the interval $[a, c]$, $\beta'(v^-) \in [-1, -0.5]$ everywhere in the interval $[a, c)$. Moreover, $\beta'(v^-)v' + b' \leq \beta'(v^-)v + b$ for any $b' \leq \beta(v')$ for all $v \in (\underline{v}, c)$, because β is concave in that interval.

Consider now the following mechanism (Q, T). For $v \leq a$, $Q(v, b) = 0$, $T(v, b) = 0$ for every $b \in [\underline{b}, \bar{b}]$. For $v \geq c$, $Q(v, b) = \lim_{v' \to c} \beta'(v^-)$ and $T(v, b) = Q(v, b)c$. For $v \in (a, c)$, $Q(v, b) = T(v, b) = 0$ for every $b < \beta(v)$. For $v \in (a, c)$, $Q(v, b) = \beta'(v^-)$ and $T(v, b) = Q(v, b)v + \beta(v)$ for every $b \geq \beta(v)$. It is easy to check that (Q, T) satisfies all of the incentive compatibility constraints. Moreover, by construction β is the exclusion boundary of (Q, T).

We now show that $\beta'(v) = -Q(v, \beta(v))$ at all points $v \in [a, c]$, where β is differentiable and $Q(v, b) \geq 0.5$.

Fix an incentive-compatible mechanism (Q, T) and let β be its exclusion boundary. Pick a type $(v, \beta(v))$ for which $Q(v, \beta(v)) \geq 0.5$ and the derivative of β at v exists. Similar to the first part of the proof, let L be the set of types that are excluded from control of the firm. This set is convex, and hence the boundary β is concave. Therefore, for any type $(v', b') \in L$, $Q(v, \beta(v))v + \beta(v) \geq Q(v, \beta(v))v' + b'$. Since L is convex, and since $\beta'(v)$ exists, it has to be that $\beta'(v) = -Q(v, \beta(v))$, because the convex set L is supported with a line (hyperplane) that passes through $(v, \beta(v))$, and the slope of the line is equal to the derivative of the boundary. $\square$

Proof of Proposition 2. Let (Q, T) be an incentive-compatible mechanism. We begin by showing that the seller's payoff from (Q, T) can be written as

$$\pi = -\int_{\underline{v}}^{\bar{v}} U(v, \underline{b})f(v)dv + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b)g(b; v)db \right\} f(v)dv. \quad (5)$$

Indeed,

$$
\begin{aligned}
\pi &= \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[T(v,b) + vP(v,b)[1 - Q(v,b)] \right] g(b;v)db \right\} f(v)dv \\
&= \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[-U(v,b) + (b+v)P(v,b) \right] g(b;v)db \right\} f(v)dv \\
&= \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[-U(v,\underline{b}) - \int_{\underline{b}}^{b} P(v,x)dx + (b+v)P(v,b) \right] g(b;v)db \right\} f(v)dv \\
&= -\int_{\underline{v}}^{\bar{v}} U(v,\underline{b})f(v)dv + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] P(v,b)g(b;v)db \right\} f(v)dv,
\end{aligned}
$$

where the second inequality follows by the definition of U, the third from Lemma 2, and the fourth from interchanging the order of integration.

Next, we rewrite the seller's profit by using screening along v. For that purpose we use the equality, which is obtained using standard analysis as in Myerson (1981),

$$
U(v,\underline{b}) = U(\underline{v},\underline{b}) + \int_{\underline{v}}^{v} P(x,\underline{b})Q(x,\underline{b})dx \tag{6}
$$

for all v.

Using (5), the seller's payoff can be rewritten as

$$
\begin{aligned}
\pi &= -\int_{\underline{v}}^{\bar{v}} U(v,\underline{b})f(v)dv + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] P(v,b)g(b;v)db \right\} f(v)dv \\
&= -U(\underline{v},\underline{b}) + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] P(v,b)g(b;v)db - P(v,\underline{b})Q(v,\underline{b})\frac{1 - F(v)}{f(v)} \right\} f(v)dv,
\end{aligned}
$$

where the second equality follows by plugging (6) into (5) and integrating by parts. $\qquad \square$

Lemma 4. *Suppose Assumption 1 is satisfied. Let $(\tilde{Q}, \tilde{T})$ be an incentive-compatible mechanism, and $\tilde{\beta}$ its corresponding exclusion boundary defined by $\tilde{\beta}(v) = \inf\{b : \tilde{P}(v,b) = 1\}$ for all $v \in [\underline{v}, \bar{v}]$. If v is such that $\tilde{\beta}(v) < \alpha(v)$, then $\tilde{\beta}(v') \leq \alpha(v')$ for all $v' > v$.*

Proof. β is absolutely continuous because, it is continuous, almost everywhere differentiable on $[\underline{v}, c]$ and its derivative is bounded whenever it exists. Therefore, we can write $\beta(v) = \beta(\underline{v}) + \int_{\underline{v}}^{v} \beta'(x)dx$. Similarly, $\alpha(v) = \alpha(\underline{v}) + \int_{\underline{v}}^{v} \alpha'(x)dx$. Let $\hat{v}$ be such that $\tilde{\beta}(\hat{v}) < \alpha(\hat{v})$. Then $\alpha(v) - \tilde{\beta}(v) = \alpha(\hat{v}) - \tilde{\beta}(\hat{v}) + \int_{\hat{v}}^{v}[\alpha'(x) - \tilde{\beta}'(x)]dx$. Let $\tilde{v} \equiv \min\{v' \geq v : \tilde{\beta}(v') = \underline{b}\}$. On the interval $[v, \tilde{v}]$, $\tilde{\beta}'(x) \in [-1, -0.5]$ and $\alpha'(x) \geq -0.5$; therefore, $\alpha(v) - \tilde{\beta}(v) = \alpha(\hat{v}) - \tilde{\beta}(\hat{v}) + \int_{\hat{v}}^{v}[\alpha'(x) - \tilde{\beta}'(x)]dx \geq 0$. On the other hand, incentive compatibility of $(\tilde{Q}, \tilde{T})$ implies $\tilde{\beta}(v) = \underline{b}$ for all $v \in [\tilde{v}, \bar{v}]$. Namely, if $\tilde{\beta}(v) = \underline{b}$, then

$\tilde{\beta}(v') = \underline{b}$ for all $v' \geq v$. Since both α and $\tilde{\beta}$ take values in $[\underline{b}, \bar{b}]$, it cannot be the case that $\alpha(v) < \tilde{\beta}(v)$ for $[\tilde{v}, \bar{v}]$, which concludes the proof. $\qquad\square$

Proof of Proposition 3. The seller is maximizing

$$\pi = -U(\underline{v}, \underline{b}) + \int_{\underline{v}}^{\bar{v}} \left\{ \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db - P(v, \underline{b}) Q(v, \underline{b}) \frac{1 - F(v)}{f(v)} \right\} f(v) dv$$

over incentive-compatible and individually rational mechanisms.

Let (Q, T) be some incentive-compatible and individually rational mechanism. Define

$$\pi(v) \equiv \int_{\underline{b}}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] P(v, b) g(b; v) db - P(v, \underline{b}) Q(v, \underline{b}) \frac{1 - F(v)}{f(v)}$$

for all v. Proposition 2 implies $\pi = -U(\underline{b}, \underline{v}) + \int_{\underline{v}}^{\bar{v}} \pi(v) f(v) dv$.

Let $(\tilde{Q}, \tilde{T})$ be an arbitrary IC and IR mechanism with an exclusion boundary $\tilde{\beta}$. We will show there exists an alternative mechanism that takes the form of a take-it-or-leave-it offer for half of the company and yields to the seller at least as high a profit as $(\tilde{Q}, \tilde{T})$ does.

Due to Lemma 4, in the Appendix, the analysis can be divided into three cases. In the first, β is never below α; in the second β crosses α from above in the interior of $[\underline{v}, \bar{v}] \times [\underline{b}, \bar{b}]$; and in the third, β is never above α.

Case 1. $\tilde{\beta}(v) \geq \alpha(v)$ for all v.

This is the case where $\tilde{\beta}$ is never below α. Two additional cases are to be considered here, depending on whether there exists a $v \in [\underline{v}, \bar{v}]$ such that $\tilde{\beta}(v) = \underline{b}$. We first consider the former case, the latter is elaborated on at the end of Case 1.

Let $v_\alpha = \inf\{v : \alpha(v) = \underline{b}\}$ and $\tilde{v} = \inf\{v : \tilde{\beta}(v) = \underline{b}\}$. In other words, v_α is the smallest v at which $\alpha(v)$ hits $\underline{b}$. Similarly, $\tilde{v}$ is the lowest v at which $\tilde{\beta}$ hits $\underline{b}$. Given the assumption of Case 1, $v_\alpha \leq \tilde{v}$.[18]

We now describe the alternative mechanism, (Q^*, T^*), which is a take-it-or-leave-it offer for half of the company at the price $\tilde{v}/2 + \underline{b}$. The boundary β^* corresponding to this

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[18]Given that we are considering the case where there exists a v such that $\tilde{\beta}(v) = \underline{b}$, both v_α and $\tilde{v}$ are well-defined.

mechanism is given by

$$\beta^*(v) = \begin{cases} \bar{b}, & \text{if } \underline{b} + \frac{\tilde{v}-v}{2} > \bar{b} \\ \underline{b} + \frac{\tilde{v}-v}{2} & \text{if } \underline{b} \leq \underline{b} + \frac{\tilde{v}-v}{2} \leq \bar{b} \\ \underline{b}, & \text{if } \underline{b} + \frac{\tilde{v}-v}{2} < \underline{b}, \end{cases}$$

for all v. We can now write (Q^*, T^*) as $Q^*(v, b) = 0.5 * 1_{[(v,b)\geq(v,\beta^*(v))]}$, and $T^*(v, b) = (v^*/2 + \underline{b}) * 1_{[(v,b)\geq(v,\beta^*(v))]}$; therefore $P^*(v, b) = 1_{[(v,b)\geq(v,\beta^*(v))]}$. Clearly this mechanism is incentive-compatible and individually rational. In the following development we will show that (Q^*, T^*) yields at least as high a payoff to the seller as $(\tilde{Q}, \tilde{T})$.

Let $v^* = \inf\{v : \beta^*(v) = \underline{b}\}$. Then $v^* = \tilde{v}$. Notice that $\alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$ for all $v \in [\underline{v}, \bar{v}]$. The first inequality holds because $v_\alpha \leq v^*$ and β^* is at least as steep (and decreasing) as α at any v such that $\beta^* \in (\underline{b}, \bar{b})$. More precisely, at any v such that $\beta(v) \in (\underline{b}, \bar{b})$, $(\beta^*)'(v) = -0.5$, while $\alpha'(v) \geq -0.5$ for all v. The inequality is then proven the same way as Lemma 4. The other inequality, $\beta^* \leq \tilde{\beta}$, follows from $v^* = \tilde{v}$ and the fact that $\tilde{\beta}$ is at least as steep (and decreasing) as β^*.

In what follows we compare the seller's profit from mechanisms $(\tilde{Q}, \tilde{T})$, denoted by $\tilde{\pi}$, with the profit from (Q^*, T^*), denoted by π^*. In fact, we will show $\pi^*(v) \geq \tilde{\pi}(v)$, which together with the fact that type $(\underline{v}, \underline{b})$ gets utility 0 in (Q^*, T^*) yields the desired result.

For $v \in [\underline{v}, \tilde{v})$,

$$\pi^*(v) - \tilde{\pi}(v) = \int_{\beta^*(v)}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)}\right] g(b; v) dx - \int_{\tilde{\beta}(v)}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)}\right] g(b; v) dx$$

$$= \int_{\beta^*(v)}^{\tilde{\beta}(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)}\right] g(b; v) dx$$

$$\geq 0,$$

where the first equality uses the fact that $\tilde{P}(v, \underline{b}) = P^*(v, \underline{b}) = 0$ on the specified range of v. The inequality follows from $\alpha \leq \beta^* \leq \tilde{\beta}$ and the fact that $v + b - \frac{1-G(b;v)}{g(b;v)}$ is nonnegative for $b \geq \alpha(v)$.

For $v \in (\tilde{v}, \bar{v}]$, $\tilde{\beta}(v) = \beta^*(v) = \underline{b}$; therefore,

$$\pi^*(v) - \tilde{\pi}(v) = [\tilde{Q}(v, \underline{b}) - Q^*(v, \underline{b})]\frac{1 - F(v)}{f(v)}$$

$$= [\tilde{Q}(v, \underline{b}) - 1/2]\frac{1 - F(v)}{f(v)}$$

$$\geq 0$$

for all v, where the last inequality follows from $\tilde{\beta}(v) = \underline{b}$ for $v \in (\tilde{v}, \bar{v}]$; therefore, $\tilde{Q}(v, \underline{b}) \geq 1/2$ for $v \in (\tilde{v}, \bar{v}]$. This concludes the analysis of the first subcase of Case 1.
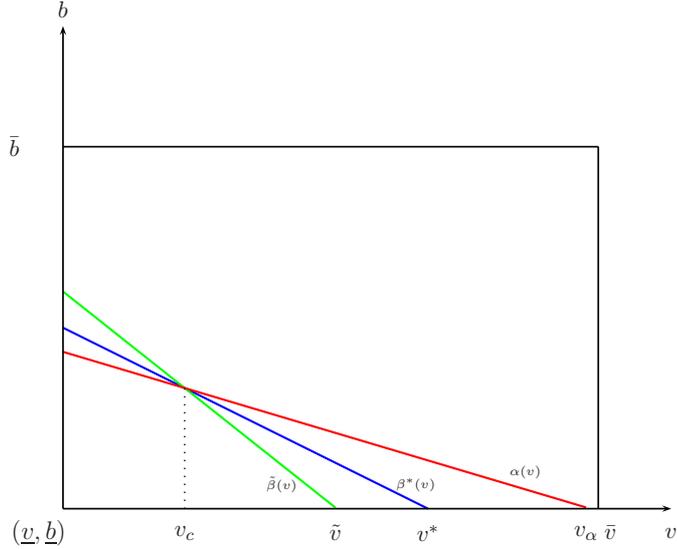
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Figure 3: This figure depicts the alternative mechanism in Case 2. The slope of the exclusion boundary of the alternative mechanism is -0.5. Note that β^* is closer to the zero virtual valuation curve α and hits the $\underline{b}$ at $v^* \geq \tilde{v}$.

We are left to consider the case where $\tilde{\beta} > \underline{b}$ for all v. In this case one can show that a take-it-or-leave-it offer for half of the company at the price $\bar{v}/2 + \tilde{\beta}(\bar{v})$ does at least as well for the seller as the mechanism $(\tilde{Q}, \tilde{T})$. The proof is very similar to the proof of the above considered case and is therefore omitted.

Case 2. There exists a v_c such that $\underline{b} < \alpha(v_c) = \tilde{\beta}(v_c) < \bar{b}$. Case 2 covers the environments in which, roughly speaking, $\tilde{\beta}$ crosses α in the interior of $[\underline{v}, \bar{v}] \times [\underline{b}, \bar{b}]$.

Fix some v_c such that $\underline{b} < \alpha(v_c) = \tilde{\beta}(v_c) < \bar{b}$. Let (Q^*, T^*) be a direct mechanism corresponding to the take-it-or-leave-it offer for half of the firm at the price of $\alpha(v_c) + v_c/2$. The corresponding β^* is

$$
\beta^*(v) = \begin{cases}
\bar{b}, & \text{if } \bar{b} < \alpha(v_c) + [v_c - v]/2 \\
\alpha(v_c) + [v_c - v]/2, & \text{if } \underline{b} \leq \alpha(v_c) + [v_c - v]/2 \leq \bar{b} \\
\underline{b}, & \text{if } \alpha(v_c) + [v_c - v]/2 < \underline{b},
\end{cases}
$$

for all v.

Due to the continuity properties of $\alpha, \tilde{\beta}, \beta^*$, and their slopes, $\alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$ for $v \leq v_c$ and $\tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$ for $v > v_c$. These inequalities can be proven following the reasoning of the proof of Lemma 4 (see also Figure 3 for a depiction).

As in Case 1, we will argue that $\pi^*(v) \geq \tilde{\pi}(v)$ for all v, which together with the fact that $U(\underline{v}, \underline{b}) = 0$ in (Q^*, T^*) implies $\pi^* \geq \tilde{\pi}$. Let $\tilde{v} = \inf\{v : \tilde{\beta}(v) = \underline{b}\}$ and $v^* = \inf\{v : \beta^*(v) = \underline{b}\}$. The fact that $\underline{b} < \beta^*(v_c) = \tilde{\beta}(v_c) < \bar{b}$ together with the fact that

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$\tilde{\beta}$ is at least as steep as β^* imply $\tilde{v} \leq v^*$. That is, for each v such that $\beta^*(v) \in (\underline{b}, \bar{b})$, $(\beta^*)'(v) = -0.5$, while for each v such that $\tilde{\beta}(v) \in (\underline{b}, \bar{b})$, $\tilde{\beta}'(v) \in [-1, -0.5]$.

For $v \leq v_c$, $\underline{b} < \alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$; therefore, $P^*(v, \underline{b}) = \tilde{P}(v, \underline{b}) = 0$. Now,

$$\pi^*(v) - \tilde{\pi}(v) = \int_{\beta^*(v)}^{\tilde{\beta}(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx$$
$$\geq 0,$$

where the inequality follows because on this range of v, $\alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$, and $v + b - \frac{1 - G(b;v)}{g(b;v)} \geq 0$ for $b \geq \alpha(v)$.

For v such that $v_c \leq v < v^*$, $\alpha(v) \geq \beta^*(v) \geq \tilde{\beta}(v) > 0$. Therefore,

$$\pi^*(v) - \tilde{\pi}(v) = -\int_{\tilde{\beta}(v)}^{\beta^*(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx + \tilde{P}(v, \underline{b})\tilde{Q}(v, \underline{b})\frac{1 - F(v)}{f(v)}$$
$$\geq 0,$$

where the last inequality follows from $v + b - \frac{1 - G(b;v)}{g(b;v)} \leq 0$ for $b \leq \alpha(v)$. The last term is 0 for $v < \tilde{v}$ and strictly positive for $v \in (\tilde{v}, v^*)$.

Finally, for v such that $v \geq v^*$,

$$\pi^*(v) - \tilde{\pi}(v) = [\tilde{Q}(v, \underline{b}) - Q^*(v, \underline{b})]\frac{1 - F(v)}{f(v)}$$
$$= [\tilde{Q}(v, \underline{b}) - 1/2]\frac{1 - F(v)}{f(v)}$$
$$\geq 0,$$

where $\tilde{Q}(v, \underline{b}) \geq 1/2$ follows from $\tilde{\beta}(v) = \underline{b}$ on the relevant range. This concludes the proof of Case 2.

Case 3. $\alpha(v) \geq \tilde{\beta}(v)$ for all v.

The analysis here divides into two additional cases depending on whether $\tilde{\beta}(\underline{v}) = \bar{b}$ or $\tilde{\beta}(\underline{v}) < \bar{b}$. We consider the first case here and the second case is simpler and similar. Let $\tilde{v}_u = \sup\{v : \tilde{\beta}(v) = \bar{b}\}$. Let (Q^*, T^*) be a take-it-or-leave-it offer for half of the firm at the price $\bar{b} + \frac{1}{2}\tilde{v}_u$. The corresponding boundary is given by

$$\beta^*(v) = \begin{cases} \bar{b}, & \text{if } \bar{b} \leq [v - \tilde{v}_u]/2 + \bar{b} \\ [v - \tilde{v}_u]/2 + \bar{b}, & \text{if } \underline{b} \leq [v - \tilde{v}_u]/2 + \bar{b} \leq \bar{b} \\ \underline{b}, & \text{if } [v - \tilde{v}_u]/2 + \bar{b} < \underline{b}. \end{cases}$$

By the definition of β^*, β^* and $\tilde{\beta}$ coincide on $[\underline{v}, \tilde{v}_u]$. On $(\tilde{v}_u, \bar{v}]$, $\tilde{\beta}(v) \leq \beta^*(v)$ because $\tilde{\beta}$ is at least as steep as β^* for all v at which $\tilde{\beta} > \underline{b}$. Likewise, $\beta^*(v) \leq \alpha(v)$ for $[\underline{v}, \tilde{v}_u]$. Therefore,

$$\tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$$

for all v. Define v^* as in Case 1.

For $v < v^*$,

$$\pi^*(v) - \tilde{\pi}(v) = -\int_{\tilde{\beta}(v)}^{\beta^*(v)} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] g(b;v) dx + \tilde{P}(v,0)\tilde{Q}(v,0)\frac{1 - F(v)}{f(v)}$$

$$\geq 0$$

because for b such that $\tilde{\beta}(v) \leq b \leq \beta^*(v) \leq \alpha(v)$, $v + b - \frac{1-G(b;v)}{g(b;v)} \leq 0$.

For $v \geq v^*$,

$$\pi^*(v) - \tilde{\pi}(v) = [\tilde{Q}(v,\underline{b}) - Q^*(v,\underline{b})]\frac{1 - F(v)}{f(v)}$$

$$= [\tilde{Q}(v,\underline{b}) - 1/2]\frac{1 - F(v)}{f(v)}$$

$$\geq 0,$$

using the same argument as in the previous cases, which concludes the proof. □

Proof of Proposition 5. Under Assumption 2, the exclusion boundary that coincides with α is implementable. Therefore, all that remains to be shown is that the seller cannot increase his profits by using any other incentive-compatible mechanism. This is straightforward because equation (2) shows that the seller's profits in any incentive-compatible mechanism are bounded above by the profits he obtains by setting $P(v, \underline{b}) = 0$, and setting $P(v, b) = 1$ only when $\phi(v, b) \geq 0$. The mechanism that implements the exclusion boundary α achieves this upper bound.

Clearly the optimal mechanism that the seller would choose if he knew v would be the one that allocates the good only if $\phi(v, b) \geq 0$. Therefore, his profits when he does not know v are identical to his profits if he knew v. □

Proof of Proposition 4. The proof is similar to the proof of Proposition 3; therefore, we will be somewhat less formal here. Let $(\tilde{Q}, \tilde{T})$ be an incentive-compatible and individually rational mechanism with an exclusion boundary $\tilde{\beta}$. We argue that there exists a take-it-or-leave-it offer for the whole company that does at least as well for the seller. Using the same argument as in the proof of Proposition 3, with the difference that here

α is steeper than $\tilde{\beta}$ in the interior of the valuation space, we can split the analysis into three cases: $\tilde{\beta}$ is never bellow α, $\tilde{\beta}$ crosses α from bellow or $\tilde{\beta}$ is never above α.

Case 1. $\tilde{\beta}(v) \geq \alpha(v)$ for all v. If $\alpha(v) < \bar{b}$ for all v, then the mechanism (Q^*, T^*) corresponding to the take-it-or-leave-it offer for the whole company at the price $\underline{v} + \alpha(\underline{v})$ does at least as well for the seller as the mechanism $(\tilde{Q}, \tilde{T})$. The boundary β^* corresponding to (Q^*, T^*) is given by

$$\beta^*(v) = \begin{cases} \alpha(\underline{v}) + \underline{v} - v, & \text{if } \alpha(\underline{v}) + \underline{v} - v \geq \underline{b}, \\ \underline{b}, & \text{if } \alpha(\underline{v}) + \underline{v} - v < \underline{b}. \end{cases}$$

It is easy to verify that the boundary β^* corresponding to (Q^*, T^*) is between α and $\tilde{\beta}$ for each v: i.e., $\alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$.

In what follows we compare the seller's profit from the mechanism $(\tilde{Q}, \tilde{T})$, denoted by $\tilde{\pi}$, with the profit from (Q^*, T^*), denoted by π^*. For the definition of $\pi(v)$ see (4). In fact we will show that $\pi^*(v) \geq \tilde{\pi}(v)$ which together with the fact that type $(\underline{v}, \underline{b})$ gets utility 0 in (Q^*, T^*) yields the desired result.

Now,

$$\begin{aligned} \pi^*(v) - \tilde{\pi}(v) &= \int_{\beta^*(v)}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx - \int_{\tilde{\beta}(v)}^{\bar{b}} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx \\ &= \int_{\beta^*(v)}^{\tilde{\beta}(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx \\ &\geq 0, \end{aligned}$$

where the first equality uses the fact that $\tilde{P}(v, \underline{b}) = P^*(v, \underline{b}) = 0$ for all v, which is due to $\underline{b} < \alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$. The inequality follows from $\alpha \leq \beta^* \leq \tilde{\beta}$ and the fact that $v + b - \frac{1 - G(b; v)}{g(b; v)}$ is nonnegative for $b \geq \alpha(v)$.

If instead $\alpha(\underline{v}) = \bar{b}$, then let $v^\alpha = \sup\{v : \alpha(v) = \bar{b}\}$. A take-it-or-leave-it offer for the whole company at the price $v^\alpha + \bar{b}$ does at least as well for the seller as the mechanism $(\tilde{Q}, \tilde{T})$. This is easily verified using the same reasoning as in the $\alpha(\underline{v}) < \bar{b}$ case.

Case 2. There exists a v_c such that $\underline{b} < \alpha(v_c) = \tilde{\beta}(v_c) < \bar{b}$, and for $v \leq v_c$ we have $\tilde{\beta}(v) \leq \alpha(v)$; for $v \geq v_c$ we have $\alpha(v) \leq \tilde{\beta}(v)$. Case 2 covers the environments in which, roughly speaking, $\tilde{\beta}$ crosses α from below in the interior of $[\underline{v}, \bar{v}] \times [\underline{b}, \bar{b}]$.

Fix some v_c such that $\underline{b} < \alpha(v_c) = \tilde{\beta}(v_c) < \bar{b}$. Let (Q^*, T^*) be a direct mechanism corresponding to the take-it-or-leave-it offer for the whole firm at the price of $\alpha(v_c) + v_c$.

The corresponding β^* is

$$\beta^*(v) = \begin{cases} \bar{b}, & \text{if } \bar{b} < \alpha(v_c) + v_c - v \\ \alpha(v_c) + v_c - v, & \text{if } \underline{b} \leq \alpha(v_c) + v_c - v \leq \bar{b} \\ \underline{b}, & \text{if } \alpha(v_c) + v_c - v < \underline{b} \end{cases}$$

for all v.

The continuity properties of $\alpha, \tilde{\beta}, \beta^*$ and their slopes imply $\tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$ for $v \leq v_c$ and $\alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$ for $v > v_c$. These inequalities can be proven following the reasoning of the proof of Lemma 4.

As in Case 1, we will argue that $\pi^*(v) \geq \tilde{\pi}(v)$ for all v, which together with the fact that $U(\underline{v}, \underline{b}) = 0$ in (Q^*, T^*) implies $\pi^* \geq \tilde{\pi}$. For $v \leq v_c$,

$$\pi^*(v) - \tilde{\pi}(v) = -\int_{\tilde{\beta}(v)}^{\beta^*(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx$$
$$\geq 0,$$

where the first equality is the consequence of $\underline{b} < \tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$ for $v \leq v_c$, and therefore $P^*(v, \underline{b}) = \tilde{P}(v, \underline{b}) = 0$. The inequality follows because on this range of v, $\tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$, and $v + b - \frac{1 - G(b; v)}{g(b; v)} \leq 0$ for $b \leq \alpha(v)$.

For $v > v_c$,

$$\pi^*(v) - \tilde{\pi}(v) = \int_{\beta^*(v)}^{\tilde{\beta}(v)} \left[v + b - \frac{1 - G(b; v)}{g(b; v)} \right] g(b; v) dx$$
$$\geq 0,$$

where the equality holds due to $\underline{b} < \alpha(v) \leq \beta^*(v) \leq \tilde{\beta}(v)$ and consequently $P^*(v, \underline{b}) = \tilde{P}(v, \underline{b}) = 0$. Notice that this is an instance where the assumption $\alpha(\bar{v}) > \underline{b}$ is used. The inequality follows from $v + b - \frac{1 - G(b; v)}{g(b; v)} \leq 0$ for $b \leq \alpha(v)$.

Case 3. $\tilde{\beta}(v) \leq \alpha(v)$ for all v. We will argue that the take-it-or-leave-it offer for the whole company at the price of $\bar{v} + \alpha(\bar{v})$ does at least as well for the seller as $(\tilde{Q}, \tilde{T})$. The boundary corresponding to the take-it-or-leave-it offer is

$$\beta^*(v) = \begin{cases} \alpha(\bar{v}) + \bar{v} - v, & \text{if } \alpha(\bar{v}) + \bar{v} - v \leq \bar{b} \\ \bar{b}, & \text{if } \alpha(\bar{v}) + \bar{v} - v > \bar{b}, \end{cases}$$

for $v \in [\underline{v}, \bar{v}]$.

Since $\beta^*(\bar{v}) = \alpha(\bar{v})$, α is at least as steep as β^*, and both are nonincreasing, $\beta^*(v) \leq \alpha(v)$ for all v. A more formal argument can be made along the lines of the argument used

in the Proof of Proposition 3. On the other hand, $\beta^*(v) = \alpha(v) \geq \tilde{\beta}(v)$, β^* being at least as steep in the interior of the type space as $\tilde{\beta}$, and both β^* and $\tilde{\beta}$ being nonincreasing yields $\tilde{\beta}(v) \leq \beta^*(v)$ for all v.

Let $\pi(v)$ be as in (4). Then

$$\pi^*(v) - \tilde{\pi}(v) = -\int_{\tilde{\beta}(v)}^{\beta^*(v)} \left[v + b - \frac{1 - G(b;v)}{g(b;v)} \right] g(b;v)dx + \tilde{P}(v,0)\tilde{Q}(v,0)\frac{1 - F(v)}{f(v)}$$
$$\geq 0,$$

where the first line follows due to $P^*(v,\underline{b}) = Q^*(v,\underline{b}) = 0$ and the inequality due to $v + b - \frac{1-G(b;v)}{g(b;v)} \leq 0$ for $b \in [\tilde{\beta}(v), \beta^*(v)]$ since $\tilde{\beta}(v) \leq \beta^*(v) \leq \alpha(v)$. This is another instance where the assumption $\alpha(\bar{v}) > \underline{b}$ is used. It enables us to construct the take-it-or-leave-it offer for the whole firm (Q^*, T^*) which does at least as well as the original mechanism $(\tilde{Q}, \tilde{T})$ in such a way that $P^*(v,\underline{b}) = Q^*(v,\underline{b}) = 0$ for all v. $\qquad\square$

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