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# Selfconfirming Equilibrium and Uncertainty*

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**Abstract**

We propose to bring together two conceptually complementary ideas: (1) selfconfirming equilibrium (SCE): at a rest point of learning dynamics in a game played recurrently, agents best respond to confirmed beliefs, i.e. beliefs consistent with the evidence they accumulate, and (2) ambiguity aversion: agents, *coeteris paribus*, prefer to bet on events with known rather than unknown probabilities, more generally, agents distinguish objective from subjective uncertainty, which is captured by their ambiguity attitudes. Using as a workhorse the "smooth ambiguity" model of Klibanoff, Marinacci and Mukerji (2005), we provide a definition of "smooth SCE" which generalizes the traditional concept of Fudenberg and Levine (1993a,b), called Bayesian SCE, and admits Waldean (maxmin) SCE as a limit case. We show that *the set of equilibria expands as ambiguity aversion increases*. The intuition is simple: by playing the same "status-quo" strategy in a stable state an agent learns the implied objective probabilities of payoffs, but alternative strategies yield payoffs with unknown probabilities; keeping beliefs fixed, increased aversion to ambiguity makes such strategies less appealing. We rely on this core intuition to show that different notions of equilibrium are nested in a simple way, from finer to coarser: Nash, Bayesian SCE, Smooth SCE and Waldean SCE. We also prove some equivalence results, under special assumptions about the information structure.

KEYWORDS: Selfconfirming equilibrium, conjectural equilibrium, uncertainty, smooth ambiguity.

JEL CLASSIFICATION: C72, D80.

*Chi lascia la via vecchia per la via nuova, sa quel che perde ma non sa quel che trova*[1]

## 1 Introduction

In a *selfconfirming equilibrium* (or conjectural equilibrium) of a game agents best respond to *confirmed* probabilistic beliefs, where "confirmed" means that their beliefs are consistent with the evidence they can collect, given the strategy that they adopt. Of course such evidence

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[1] Italian proverb "Those who leave the old road for a new one, know what they leave but do not what they will find."

depends on how everybody else plays. This is interpreted as a characterization by means of a static definition of equilibrium of the rest points of plausible adaptive learning dynamics in games played recurrently: when a game is played recurrently, agents best respond to updated beliefs given their accumulated experience, as soon as their beliefs stabilize, agents start choosing always the same strategies and a rest point is reached.

The SCE concept can be framed within different scenarios. A simple scenario is just a repeated game with a fixed set of players, where the constituent game which is being repeated may have sequential moves, and monitoring may be imperfect. To avoid repeated game effects, it is assumed that players do not value their future payoffs, they simply best respond to their updated beliefs about the current period strategies of the opponents. Here we mainly refer to a more complex scenario that is more appropriate for the ideas we want to explore: there is a large society of individuals who play recurrently a given game $\Gamma$, possibly a sequential game, possibly with chance moves; for each player/role $i$ in $\Gamma$ (male or female, buyer or seller, etc.) there is a large population of agents who play in role $i$. Agents are drawn at random and matched to play $\Gamma$, then separated and re-matched to play $\Gamma$ with (almost certainly) different co-players, and so on. After each play of a game in which he was involved, an agent obtains some evidence on how the game was played. The accumulated evidence is the data set used by this agent to update his beliefs. Note, there is an intrinsic limitation to the evidence that an agent can obtain: at most he can observe the path (terminal node) realized in the game he just played, but often he can observe even less, e.g. only his monetary payoffs. However, what each agent is really interested about is the statistical distribution of strategies in the populations corresponding to opponents' roles, as such distributions determine (*via* random matching) the objective probabilities of different strategy profiles of the opponents he is matched with. Similar considerations hold for chance moves, when their probabilities are unknown. This defines the fundamental inference problem faced by an agent.

The key difference between SCE and Nash equilibrium is that in an SCE agents may have incorrect beliefs because many possible underlying distributions are consistent with the empirical frequencies they observe. Two examples can clarify this point (see Fudenberg and Levine 1993b, Fudenberg and Kreps 1995).

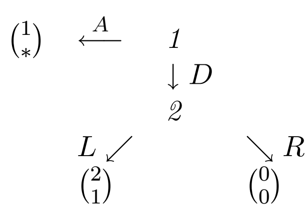


**Figure 1**

**Example 1** *In the game of Figure 1 there are two Nash equilibrium outcomes (terminal nodes), $A$ and $(D, L)$. But, assuming ex post observation of the terminal node, or of realized payoffs, there is a continuum of SCE distributions on terminal nodes of the form $\alpha A + (1 - \alpha)(D, L)$. The reason is that there may be two endogenously determined "epistemic types" of agents in population 1: a fraction $\alpha$ of "pessimistic" agents who believe that more than half of the agents in population 2 play $R$ and a fraction $(1 - \alpha)$ of agents in population 1 who believes that all agents in population 2 play $L$. The latter belief is indeed correct: all agents in* *population 2 choose their weakly dominant strategy, corresponding to a conditionally strictly dominant action, L.*[2] *The "pessimistic" agents in population 1 choose the "safe" action A (across) and obtain trivial evidence (their payoff is 1); their beliefs are necessarily consistent with the evidence they obtain. The other agents play D and every time observe ex post (directly, or indirectly via their realized payoff) that the opponent they are matched with plays L. Their confirmed belief is therefore that all agents in population 2 play L, or – equivalently - that the probability of L is* 1. *In such SCEs, the beliefs of the pessimistic agents are incorrect, but they are (trivially) consistent with their evidence.* ▲

$$
\begin{array}{cccccccccc}
 & 1 & & --- & \longrightarrow & 2 & & --- & \longrightarrow & \binom{1}{1} \\
 & | & & A & & | & & a & & \\
 & \downarrow D & & & & \downarrow d & & & & \\
 & 0 & & \ldots\ldots & \ldots\ldots & 0 & & & & \\
L \swarrow & & & \searrow R & L \swarrow & & & \searrow R & & \\
\binom{3}{3} & & & \binom{0}{0} & \binom{0}{0} & & & \binom{3}{3} & &
\end{array}
$$

**Figure 2**

**Example 2** *In the "horse" game of Figure 2, player* 0 *is Chance, or some indifferent player;*[3] *the dotted line means that Chance cannot distinguish D from $(A, d)$ and hence chooses between L and R with the same objective probabilities, which are assumed to be unknown to the agents in populations 1 and 2. Assuming that agents can observe ex post the terminal node, or their realized payoff, there is an SCE with the following features: (1) all agents of population 1 play A believing that R is very likely and that all agents in 2 play a, (2) and all agents of population 2 play a believing that L is very likely and that all agents in 1 play A. The resulting outcome is $(A, a)$. Of course, someone must be wrong, but all beliefs are consistent with the evidence, i.e. they are confirmed. On the other hand, in a Nash equilibrium beliefs about the third player (be it chance or a real player), must be correct, hence they must agree: if R is (believed) more likely, 1 and 2 play $(A, d)$; if L is (believed) more likely, 1 chooses D. There is no Nash equilibrium with outcome $(A, a)$.* ▲

According to the traditional SCE concept, agents are subjective expected utility (SEU) maximizers. But when a whole set of underlying distributions is consistent with their information, agents face a condition of uncertainty, or *ambiguity*, rather than risk. It is therefore plausible to assume that they have non-SEU attitudes toward ambiguity and decide accordingly. Many models of choice capturing non-Bayesian ambiguity attitudes have been studied in decision theory (see Gilboa and Marinacci, 2011). The decision theoretic work which is more germane to our approach distinguishes between objective and subjective uncertainty: given a set $S$ of states (pure strategies of others, in our application), there is a set $\Sigma \subseteq \Delta(S)$ of possible probabilistic "models" (strategy distributions); each $\sigma \in \Sigma$ specifies

[2] It can be shown that every SCE distribution on terminal nodes can be induced by SCE strategies that are not conditionally dominated.

[3] The only reason to have chance or an indifferent player is to simplify the picture: this way there is no need to specify the payoffs of this player. Such payoffs are immaterial for the example.

the objective probabilities of states and, for each action $a$ of the decision maker (DM), a von Neumann-Morgenstern expected utility $U(a,\sigma)$; the DM is uncertain about the true model $\sigma$.[4] According to the smooth model of Klibanoff, Marinacci, and Mukerji (2005), KMM hereafter, this uncertainty is represented by subjective beliefs $p \in \Delta(\Sigma)$. If DM is Bayesian, he evaluates each $a$ according to its SEU, by reduction of compound lotteries:

$$V(a,\Sigma,p) = \int_{\Sigma} U(a,\sigma)\mathrm{d}p,$$

but a non Bayesian DM does not carry out a simple reduction of compound lotteries, he weights the objective EUs in the set $\{u : \exists\sigma \in \Sigma, u = U(a,\sigma)\}$ according to an increasing function $\phi$ that captures his attitudes toward ambiguity:

$$V(a,\Sigma,p;\phi) = \phi^{-1}\left(\int_{\Sigma} \phi(U(a,\sigma))\mathrm{d}p\right).$$

The Bayesian value obtains when $\phi$ is linear; DM is ambiguity averse when $\phi$ is concave, and if $p$ has full support, as the concavity of $\phi$ goes to infinity the "Waldean" value

$$V(a,\Sigma) = \min_{\sigma\in\Sigma} U(a,\sigma)$$

obtains in the limit.

In our framework, $a$ is the action, or strategy, of an agent playing in role $i$, $\sigma$ is a distribution of strategies in the population of opponents (or a profile of such distributions in $n$-person games), and $\Sigma$ is the set of distributions consistent with the database of the agent. In a selfconfirming equilibrium each agent in each role best responds to his database choosing the action with the highest value, and his database is the one that obtains under the true data generating process corresponding to the actual strategy distributions. The following example shows how this notion of SCE differs from the traditional, or Bayesian, SCE.

$$\begin{array}{ccccccc}
\binom{x}{x} & \xleftarrow{A} & - & 1 & & & \\
 & & \swarrow L & & R \searrow & & \\
 & 2 & -- & -- & -- & 2 & \\
\ell \swarrow & & \searrow r & & \ell \swarrow & & \searrow r \\
\binom{3}{3} & & \binom{0}{0} & & \binom{0}{0} & & \binom{3}{3}
\end{array}$$

**Figure 3**

**Example 3** *Consider the common interest game in Figure 3: assuming perfect feedback and* $0 < x < 3/2$.[5] *It is easy to check (see Section 5) that strategy* $A$ *cannot have positive probability (i.e., it cannot be played by a positive fraction of agents) in any Bayesian SCE, and that it can be played if player 1 is sufficiently ambiguity averse. For instance, if player 1 could commit to any objective randomization device,*[6] *strategy* $A$ *is dominated by the mixed*

[4] See Cerreia-Vioglio et al (2011a,b).

[5] We specify player's 2 payoffs for definiteness and later reference, but they are immaterial for the point we make here; player 2 could be Nature, as long as its strategy distribution is unknown.

[6] We think that allowing for randomization is questionable. Nonethelss we analyze equilibria with randomization in section 6.

*strategy $\frac{1}{2}L + \frac{1}{2}R$, which yields expected payoff $3/2$ irrespective of 2's strategy; according to the Waldean criterion the value of playing L or R is zero if every distribution on player's 2 strategies is deemed possible.*

*Why should every distribution on $\{\ell, r\}$ be deemed possible? The reason is that, according to the selfconfirming equilibrium approach, if an individual in role 1 keeps playing the status-quo action/strategy A, he has no way to learn the strategy distribution of the co-player.* ▲

The example shows that the set of co-players strategy distributions deemed possible by an agent playing in role $i$ is endogenously determined by the selfconfirming equilibrium conditions as the set of strategies distributions consistent with the distribution of observations of this agent given the status-quo strategy he keeps playing and given that each agent is best responding according to a well defined decision criterion.

**Related literature** To the best of our knowledge, the papers most related to our idea of combining SCE with non-Bayesian attitudes toward uncertainty are Lehrer (2011) and Lehrer and Teper (2009). In these papers a decision maker (DM) is endowed with a "partially specified probability" (PSP), that is, a list of random variables defined on a probability space. The DM knows only the expected values of the random variables, hence he is uncertain about the true underlying probability measure within the set of all measures that give rise to such values. Lehrer (2001) axiomatizes a decision criterion related to the maximization of the minimal expected utility with respect to such set of probability measures consistent with the PSP. Then he defines a notion of equilibrium with partially specified probabilities for a game in strategic form under the assumption that each player obtains separate feedback about each opponent's strategy and this feedback is independent of the strategy he plays. Note that the latter assumption[7] is violated in all the above examples. Under these assumptions is equilibrium is similar the one we obtain in the "Waldean" case. Lehrer and Teper (2009) explicitly relate their work to the SCE literature, but the decision criterion they axiomatize for a given PSP, a kind of dual of Bewley's (2002) Knightian criterion featuring intransitivity, is quite different from those considered here.

We are not going to thoroughly review the vast literature on uncertainty and ambiguity aversion, which is covered by a comprehensive recent survey (Gilboa and Marinacci, 2011). We only mention that in the paper we rely on the decision theoretic framework of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011b) that makes formally explicit the DM's uncertainty about the true probabilistic model, or data generating process.

The literature on SCE and related ideas is smaller, but it is not covered by any comprehensive survey.[8] Hahn (1973) intuitively, but quite clearly articulates the idea that an economic system is in equilibrium when agents' strategies are best replies to beliefs confirmed by the evidence generated by the actual strategy profile. Later, Hahn called such states "conjectural equilibria," but he did not provide the game-theoretic formalization that is necessary to express his original idea properly (see Hahn 1977, 1978). Battigalli (1987) provided a precise a game theoretic definition of such equilibria, explicitly, but only intu-

[7] An assumption called "non-manipulability of strategic information" in Battigalli et al (1992).

[8] For partial surveys, see Battigalli et al (1992) and chapter 6 of Fudenberg and Levine (1998). Cho and Sargent (2008) briefly survey the macroeconomic applications of the selfconfirming equilibrium idea.

itively motivating it as characterizing rest points of learning dynamics in recurrent games.[9] Similarly to our paper, information partitions of the set of terminal nodes (complete paths) are introduced among the primitives of the framework, and it is noted that strategies in non-trivial extensive-form games cannot be perfectly observable ex post, even under the assumption of *perfect feedback*, i.e., of perfect ex post observability of the path. The simple reason is that contingent choices at off-path information sets cannot be observed. Hence, the opponents' mixed strategies consistent with observed frequencies are only partially identified, leaving players in a state of uncertainty. This makes "conjectural equilibrium" a coarser concept than Nash equilibrium, although a path-equivalence result is established for two-person games. Battigalli (1987) studied conjectural equilibria satisfying two properties: (i) *unitary beliefs,* all the pure strategies in the support of a player's mixed strategy are best replies to the same beliefs about others, and (ii) such beliefs satisfy stochastic *independence*. Although worth exploring, such properties are not warranted by a rigorous learning analysis, which was first provided in independent work by Fudenberg, Kreps and Levine (see in particular Fudenberg and Kreps 1995, and Fudenberg and Levine, 1993a,b). While exploring the learning foundations for the Nash equilibrium concept, these authors realized that non-Nash states – called "selfconfirming equilibria" – can be stationary and also asymptotically stable under learning. Roughly, what they call "selfconfirming equilibria" correspond to Battigalli's conjectural equilibria under the perfect feedback assumption, once the unitary belief and independence assumptions are dropped. Since then "selfconfirming equilibrium" has been used for models with perfect feedback, and "conjectural equilibrium" for models where feedback may be imperfect.[10] But this is like using different names for equilibria of games with different rules, as old oligopoly theory used to do. We think a unified terminology is called for, and adopt here the more self-explanatory "selfconfirming equilibrium" (SCE).

Kalai and Lehrer (1993a,b, 1995) analyzed a similarly motivated notion of equilibrium, called "subjective equilibrium," in the context of repeated games, where players have to take into account the impact of their current actions on the opponents' future behavior. They showed that when players best respond to beliefs about their opponents' repeated-game strategies and if beliefs and strategies are such that players cannot be "surprised," then continuation-strategies and beliefs converge to a subjective equilibrium, which turns out to be is realization-equivalent to a Nash equilibrium under perfect monitoring. Eyster and Rabin (2005), Jehiel (2005), Esponda (2008) and Azrieli (2009) proposed equilibrium notions that can be interpreted as refinements of SCE based on simplicity of agents' subjective models of others. Fudenberg and Levine (2006) analyze a refinement of SCE (for a simple class of perfect information games) based on the idea of steady state learning with extremely patient and long-lived agents. Battigalli (1987) refined the set of SCE by looking at those that, intuitively, satisfy a form of common belief in extensive-form rationality (see also Battigalli and Guaitoli, 1988, 1997). More interestingly, and independently, Rubinstein and Wolinsky (1994) analyzed *rationalizable* SCE (in static games), i.e. states where (1) players are rational, (2) their beliefs are confirmed (but possibly incorrect), and there is common belief of (1) *and* (2).[11] Similar ideas have been explored in the context of extensive-form

[9] The first work in English on this notion of equilibrium is Battigalli and Guaitoli (1988), eventually published in 1997.

[10] An exception to this terminology is Fudenberg and Kamada (2011): they analyze a generalization of Battigalli's conjectural equilibrium that allows for correlated beliefs and call it "partition confirmed equilibrium."

[11] In static games, this refines Battigalli's (1987) equilibria, who only rest on the assumptions of rationality,

games by Dekel et al (1999), Battigalli (1999), Fudenberg and Kamada (2011), and in the context incomplete-information games by Battigalli and Siniscalchi (2003), Esponda (2011) and Letina (2011).[12]

The rest of the paper is structured as follows. In Section 2, after a few preliminaries, we start describing the main idea in recurrent decision problems where the state of nature has an unknown distribution. In Section 2 we introduce some key concepts in a decision theoretic context. Then we analyze games: we propose definitions of selfconfirming equilibrium with ambiguity attitudes (Section 3.2) and we carry out some key comparative statics exercises (Section 4). After an illustrative example (Section 5), we explore the consequences of allowing commitment to objective randomization devices (Section 6). We conclude with a detailed discussion of some important theoretical issues (Section 7).

# 2 Recurrent decisions

We first introduce our concepts in the context of a recurrent decision problem where the outcome of the decision maker's (DM) strategy depends on the state (or strategy) of nature.

## 2.1 Mathematics

Given any measurable space $(X, \mathcal{X})$, we denote by $\Delta(X)$ the collection of all probability measures $\nu : \mathcal{X} \to [0,1]$. When $X$ is finite, say with cardinality $n$, we assume that $\mathcal{X} = 2^X$ and we identify $\Delta(X)$ with the simplex of $\mathbb{R}^n$.

We endow $\Delta(X)$ with the smallest $\sigma$-algebra that makes the real valued and bounded functions on $\Delta(X)$, of the form $\nu \mapsto \nu(E)$, measurable for all $E \in \mathcal{X}$. When $X$ is finite this $\sigma$-algebra coincides with the relative Borel $\sigma$-algebra that $\Delta(X)$ inherits as the simplex of $\mathbb{R}^n$. Finally, we also endow any measurable subset $\Sigma$ of $\Delta(X)$ with the relative $\sigma$-algebra, inherited from $\Delta(X)$, and we denote by $\Delta(\Sigma)$ the collection of all probability measures defined on such $\sigma$-algebra. Among them, $\delta_x$ denotes the Dirac measure concentrated on $x \in X$, i.e., $\delta_x(x) = 1$.

Given $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$, a pair of measurable spaces, we endow the Cartesian product $X \times Y$ with the product $\sigma$-algebra $\mathcal{X} \times \mathcal{Y}$. For example, $\mathcal{X}$ and $\mathcal{Y}$ are either finite or Borel $\sigma$-algebras of separable metric spaces. We denote by $\Delta(X) \otimes \Delta(Y)$ the collection of all *product probability measures*. Moreover, each measurable function $\varphi : X \to Y$ induces the pushforward map $\hat{\varphi} : \Delta(X) \to \Delta(Y)$ defined by

$$\hat{\varphi}(\nu) = \nu \circ \varphi^{-1} \qquad \forall \nu \in \Delta(X).$$

In other words, we have that $\hat{\varphi}(\nu)(E) = \nu(\varphi^{-1}(E))$ for all $E \in \mathcal{Y}$.

Finally, $\Delta^*(X)$ denotes the collection of all affine functions $\varphi : \Delta(X) \to \mathbb{R}$, that is, $\varphi(\alpha\nu + (1-\alpha)\nu') = \alpha\varphi(\nu) + (1-\alpha)\varphi(\nu')$ for all $\alpha \in [0,1]$ and all $\nu, \nu' \in \Delta(X)$.

---

confirmed beliefs, and common belief in rationality (but not in confirmed beliefs).

[12] Esponda and Letina provide rigorous, epistemic and algorithmic characterizations rationalizable selfconfirming equilibrium. Battigalli (1999) and Battigalli and Siniscalchi (2003) also provide algorithmic characterizations, but they refer to the extensive-form epistemic analysis of Battigalli and Siniscalchi (1999, 2002) for a rigorous epistemic characterization, which is therefore not provided explicitly.

## 2.2 The decision problem

We assume there is a large population of DMs.[13] In each period, each of them faces the same decision problem which consists of a finite state space $S_N$, a finite strategy space $S_{DM}$, and a finite outcome space $Z$. This may be a decision problem in extensive form representable by a decision tree with moves by the DM and moves by Nature. In this case $S_i$, with $i \in \{DM, N\}$, is the set of pure strategies of $i$. In this paper we do not address dynamic consistency issues: we assume that the DM commits to a particular strategy that is automatically implemented in the decision problem. Hence $S_{DM}$ is the set of *actions* which DMs can choose from and, to ease notation, we write $A$ in place of $S_{DM}$ and $\Omega$ in place of $S_N$.

An *outcome function*

$$\zeta : A \times \Omega \to Z$$

maps actions and states to "outcomes". The set of outcomes $Z$ should be interpreted as the set of terminal nodes of the decision tree.[14] Such terminal nodes give rise to consequences $c \in C$ according to a function[15]

$$\gamma : Z \to C.$$

Ex post, DMs receive a message $m \in M$ according to a *feedback function*

$$f : Z \to M.$$

The function $f$ maps outcomes to messages. A DM that gets message $m$ knows that the realized outcome belongs to the set $f^{-1}(m) \subseteq Z$.

Together the outcome and the feedback functions determine the *message function*

$$F : A \times \Omega \to M$$

given by $F = f \circ \zeta$, that is, $F(a, \omega) = f(\zeta(a, \omega))$ for all action/state profiles $(a, \omega) \in A \times \Omega$. Let $F_a : \Omega \to M$ be the section at strategy $a$ of $F$ defined by $F_a(\omega) = F(a, \omega)$ for all $\omega \in \Omega$. When each $F_a$ is one-to-one, under each action different states of Nature generate different messages. Thus, ex post DMs learn the true state upon receiving the message. This, however, can happen when two conditions are satisfied: (i) the section $\zeta_a$ is one-to-one, i.e., in the decision tree Nature has no move following a move by DM, and (ii) the feedback function $f$ is one-to-one, i.e., there is *perfect feedback*. Note that under perfect feedback it is wlog to set $M = Z$ and $f = \mathrm{Id}_Z$.

Elements $\sigma$ of $\Delta(\Omega)$ are interpreted as the vectors of "objective" probabilities that each state $\omega$ obtains, that is, as possible probabilistic *models* for states. If the decision problem is viewed as a game with Nature then $\sigma$ is a mixed strategy of Nature. Thus, the true model can be seen as Nature's actual mixed play.

[13] The existence of many DMs plays an important role only in our game-theoretic analysis. Nevertheless, in order to be consistent, we mention it also in this section even though we mainly focus on a specific DM.

[14] Thus, this is the same as the "outcome function" in chapter 6 of Osborne and Rubinstein (1994), generalized to games with imperfect information.

[15] We could simplify the notation identifying terminal nodes and consequences, but we feel that introducing the consequence function explicitly improves conceptual clarity.

Elements $\alpha$ of $\Delta(A)$ can be seen as mixed actions implemented by a random device,[16] or – in our preferred interpretation – as statistical distributions of actions in the population of DMs. In both cases, $\alpha(a)$ is the "objective" probability that the DM (or a DM chosen at random) selects $a$.

A distribution on $A \times \Omega$, in particular a product distribution $\alpha \times \sigma$, delivers a random message. Specifically, each message function $F : A \times \Omega \to M$ induces a mixed message function $\hat{F} : \Delta(A) \otimes \Delta(\Omega) \to \Delta(M)$, where

$$\hat{F}(\alpha \times \sigma)(m) = (\alpha \times \sigma)\left(F^{-1}(m)\right)$$

is the probability that a DM observes an ex post message $m$ given the product measure $\alpha \times \sigma$ of a mixed action $\alpha$ and a model $\sigma$. We will mostly take the point of view of a DM who chooses a pure action $a$, but we will also consider the case of randomization, hence our general notation. We denote by $\mu$ a generic mixed message in $\Delta(M)$.

## 2.3 Criteria

The stage decision problem features two sources of uncertainty: state uncertainty within each model and model uncertainty across different models.

As to state uncertainty, given a model $\sigma$ the DMs evaluate actions $a$ through their expected utility

$$U(a, \sigma) = \sum_{\omega \in \Omega} u(\zeta(a, \omega))\, \sigma(\omega) \tag{1}$$

where $u = v \circ \gamma : Z \to \mathbb{R}$ and $v : C \to \mathbb{R}$ is a von Neumann-Morgenstern (vNM) utility function, which captures the attitudes toward risk of the DM.[17] Our main results rely on the assumption that payoffs are *observable*: the payoff function $u$ is $f$-measurable, that is, $f(z) = f(z')$ implies $u(z) = u(z')$ for all $z, z' \in Z$.

As to model uncertainty, let $U(a, \cdot) : \Delta(\Omega) \to \mathbb{R}$ be the affine function that, given an action $a$, associates to each model $\sigma$ its expected utility (1). Suppose that, based on his information, the DM knows that the true model $\sigma$ belongs to a compact subset $\Sigma \subseteq \Delta(\Omega)$. In other words, he is able to posit a set $\Sigma$ of possible models, a standard assumption in classical statistics. The payoff scope of model uncertainty for the DM that chooses action $a$ is thus described by the expected utility profile $\{U(a, \sigma) : \sigma \in \Sigma\}$.

The simplest criterion to deal with both state and model uncertainty is the *Waldean criterion*, which ranks actions according to

$$V(a, \Sigma) = \min_{\sigma \in \Sigma} U(a, \sigma). \tag{2}$$

This criterion, due to Wald (1950), does not assume any knowledge on models other than what is necessary to posit $\Sigma$. In particular, no prior probability appears. In contrast, the

[16] Random devices (e.g., roulette wheels and dice) feature probabilities that can be computed according to Laplace's classical notion.

[17] For example, if $\gamma : Z \to [\underline{r}, \bar{r}] \subseteq \mathbb{R}$ specifies monetary payoffs and $u$ is concave then the DM is risk averse.

*classical Bayesian criterion*[18] ranks actions according to

$$V(a,p,\Sigma) = \int_{\Sigma} U(a,\sigma)\, dp(\sigma) \tag{3}$$

where $p \in \Delta(\Sigma)$ is a prior probability on the possible models, a subjective probability that quantifies the DM personal information on models. Set

$$\bar{p}(\cdot) = \int_{\Sigma} \sigma(\cdot)\, dp(\sigma).$$

The reduced probability $\bar{p} \in \Delta(\Omega)$ is the *predictive probability* on states induced by the prior $p$. Using the predictive probability, we can write

$$\int_{\Sigma} U(a,\sigma)\, dp(\sigma) = \sum_{\omega \in \Omega} u(\zeta(a,\omega))\, \bar{p}(\omega) = U(a,\bar{p}).$$

The predictive form $U(a,\bar{p})$ is a "reduced form" of (3). Note that $\bar{p}$ is a subjective probability, so that $U(a,\bar{p})$ is a subjective expected utility criterion a la Savage (1954), while $U(a,\sigma)$ was a vNM one.

DMs may well be ambiguity averse, that is, they may dislike having a range of possible vNM expected utilities $\{U(a,\sigma) : \sigma \in \Sigma\}$ that is not a singleton because of model uncertainty. For this reason the *classical smooth ambiguity* criterion ranks actions according to

$$V(a,p,\Sigma;\phi) = \phi^{-1}\left(\int_{\Sigma} \phi(U(a,\sigma))\, dp(\sigma)\right) \tag{4}$$

where $p \in \Delta(\Sigma)$ is a prior probability on $\Sigma$ and $\phi : \operatorname{Im} U \to \mathbb{R}$ is a strictly increasing continuous function that describes ambiguity attitudes.[19] In particular, a concave $\phi$ captures ambiguity aversion, while a linear $\phi$ corresponds to ambiguity neutrality. The smooth criterion thus relaxes the assumption that agents linearly combine attitudes toward state and toward model uncertainty, which are allowed to differ.

The Waldean and classical Bayesian criteria can be viewed as, respectively, a limit case and a special case of the classical smooth criterion. For, under ambiguity neutrality the smooth model reduces to the Bayesian one (3), while under extreme ambiguity aversion – that is, when ambiguity aversion "goes to infinity" – the smooth model reduces to the limit to the Wald criterion (2) provided that $\operatorname{supp}(p) = \Sigma$. We refer the reader to KMM for details. Note that the Bayesian criterion should be written as $V(a,p,\Sigma;\mathrm{Id}_{\mathbb{R}})$ as a special smooth criterion; to ease notation in what follows we will keep writing $V(a,p,\Sigma)$.

We will briefly consider the possibility that a DM can delegate his choice to a random device, thus implementing a mixed strategy $\alpha \in \Delta(A)$. All the formulas above can be adapted, replacing $a$ with $\alpha$ and $U(a,\sigma)$ with the vNM expected utility

$$U(\alpha,\sigma) = \sum_{a \in A} \alpha(a) U(a,\sigma).$$

[18] See Cerreia-Vioglio et al (2011b) for a derivation of this model and a discussion of its relations with Savage (1954). The adjective "classical" here and for the smooth model below is due to the classical statistics assumption of a class $\Sigma$ of possible models.

[19] The criterion is due to Klibanoff, Marinacci, and Mukerji (2005). With a slight abuse of notation, here $\operatorname{Im} U$ is the smallest interval that contains $\bigcup_{a \in A} U(a,\cdot)$, i.e., its convex hull.

## 2.4 Recurrent decisions and status quo

The decision problem is faced recurrently by a large population of DMs. In each period one of them is drawn at random and faces the stage decision problem. In most of the paper we assume that DMs select pure actions $a$. Mixed actions $\alpha$ are interpreted as population *action distributions*, that is, $\alpha(a)$ is the fraction of DMs who choose $a$ when selected for the stage decision problem. In a classical approach a la Laplace, this fraction is the "objective" probability that a DM drawn at random chooses $a$. Formally, action distributions and mixed actions are identical, in the spirit of the Nash mass action interpretation (see, e.g., Weibull, 1996). In Section 6 we will also consider the case where a DM can play a mixed strategy $\alpha$, therefore our general formulas encompass this case too.

To abstract away from learning issues, we interpret mixed action also as the empirical frequency of actions actually chosen by DMs, that is, $\alpha(a)$ is also the long run frequency with which $a$ is chosen by the DMs who have been drawn for the stage decision problem. In a frequentist approach a la von Mises, this frequency is the "objective" probability that a DM drawn at random chooses $a$.[20] The probabilities that $\alpha$ features thus have a dual, classical and frequentist, interpretation as both population distributions and empirical frequencies. Such duality relies, heuristically, on a "long run" that is long and stationary enough for asymptotic ergodic-type results to hold.

A similar dual interpretation applies to $\sigma$, where $\sigma(\omega)$ is both the classical probability with which Nature (regarded as a random device) selects state $\omega$ at each stage and the empirical frequency of such state.

In view of all this, mixed messages $\mu$ of $\Delta(M)$ can be interpreted as long run frequency distributions of messages received by DMs, so that $\mu(m)$ is the empirical frequency of message $m$. In particular, the pushforward map $\hat{F} : \Delta(A) \otimes \Delta(\Omega) \to \Delta(M)$ associates to each profile $(\alpha, \sigma)$ the message distribution

$$\hat{F}(\alpha \times \sigma)(m) = (\alpha \times \sigma)\left(F^{-1}(m)\right) \qquad \forall m \in M$$

which is the probability that DMs observe message $m$ given an action distribution $\alpha$ and a model $\sigma$. Because of the dual nature of $\alpha$ and $\sigma$, such probability is both the "objective" probability that each stage generates message $m$ and the empirical frequency with which DMs receive such message.

Consider the section $\hat{F}_\alpha : \Delta(\Omega) \to \Delta(M)$ defined by $\hat{F}_\alpha(\sigma) = \hat{F}(\alpha \times \sigma)$. Its inverse correspondence $\hat{F}_\alpha^{-1}$ partitions $\Delta(\Omega)$ in equivalence classes

$$\hat{F}_\alpha^{-1}(\mu) = \left\{\sigma \in \Delta(\Omega) : \hat{F}(\alpha \times \sigma) = \mu\right\}$$

of models that are observationally equivalent given that $\alpha$ is played and the frequency distribution of messages $\mu$ is "observed" in the long run. In other words, $\hat{F}_\alpha^{-1}(\mu)$ is the collection of all models that may have generated $\mu$ given $\alpha$. The DMs regard $\hat{F}_\alpha^{-1}(\mu)$ as the set of models consistent with evidence $\mu$ given $\alpha$; for this reason, we say that they are *long run empiricists*.

[20] This frequentist interpretation actually requires that the stage decisions be independent (an "ergodic" interpretation of probabilities as time averages hold, however, more generally; see Lith, 2001, and the references therein).

The inverse correspondence is compact valued. It is a function if and only if $\hat{F}_\alpha$ is one-to-one; in this case the decision problem is identified under $\alpha$. Different models generate different message distributions which thus uniquely pin down models. It is the counterpart in our setup of the classic notion of identifiability (see, e.g., Rothenberg, 1971). Accordingly, we only have *partial identification* when $\hat{F}_\alpha$ is not one-to-one. In the extreme case when $\hat{F}_\alpha$ is constant – that is, all models generate the same message distribution – the decision problem is completely unidentified.

For each $\alpha \in \Delta(A)$, consider the correspondence

$$\tau_\alpha = \hat{F}_\alpha^{-1} \circ \hat{F}_\alpha : \Delta(\Omega) \to 2^{\Delta(\Omega)}$$

For any fixed $\sigma^* \in \Delta(\Omega)$, the image

$$\tau_\alpha(\sigma^*) = \left\{\sigma \in \Delta(\Omega) : \hat{F}(\alpha \times \sigma) = \hat{F}(\alpha \times \sigma^*)\right\} \tag{5}$$

is the collection of models that are observationally equivalent given $\alpha$ and the message distribution $\mu = \hat{F}(\alpha \times \sigma^*)$ that $\alpha$ generates along with model $\sigma^*$.

We can thus regard $\tau_\alpha$ as the *identification correspondence* determined by $\alpha$. It is easily seen to be compact valued and nonempty (since $\sigma \in \tau_\alpha(\sigma)$). It is a function if and only if $\hat{F}_\alpha$ is one-to-one. In this case, $\tau_\alpha$ is the identity function, with $\tau_\alpha(\sigma) = \sigma$ for all $\sigma \in \Delta(\Omega)$, and so message distributions identify the true model.

Given a mixed strategy $\alpha$, its identification correspondence partitions $\Delta(\Omega)$ in sets $\tau_\alpha(\sigma)$ of observationally equivalent models. Next we show that, under the observable payoff assumption, such models share the same expected utility, that is, the function $U(\alpha, \cdot) : \Delta(\Omega) \to \mathbb{R}$ is measurable with respect to the partition determined by $\tau_\alpha$.

**Lemma 4** *Under observable payoffs, given any $\sigma \in \Delta(\Omega)$ and any $\alpha \in \Delta(A)$, it holds*

$$U(\alpha, \sigma') = U(\alpha, \sigma) \qquad \forall \sigma' \in \tau_\alpha(\sigma).$$

In the lemma a key status quo bias emerges. For, suppose that the agent keeps playing, for example, a pure action $a$ and that $\sigma$ is the true model, so that $\tau_a(\sigma)$ is the relevant evidence for his stage decision. By the lemma, the function $U(a, \cdot)$ is constant on $\tau_a(\sigma)$, that is, the agent does not perceive any model uncertainty in his evaluation of the "status quo" action $a$. In contrast, model uncertainty may affect the evaluation of any other action $a'$ since the function $U(a', \cdot)$ might well be nonconstant on $\tau_a(\sigma)$. This status quo bias will play a key role in Theorem 9, the paper main result.

**Proof** Since payoffs are observable, there exists a function $\bar{u} : \operatorname{Im} f \to \mathbb{R}$ such that $u = \bar{u} \circ f$. For each $\sigma' \in \Delta(\Omega)$ it holds

$$\begin{aligned} U(\alpha, \sigma') &= \sum_{a \in A} \sum_{\omega \in \Omega} (\bar{u} \circ f)(\zeta(a, \omega))\, \sigma'(\omega)\, \alpha(a) \\ &= \sum_{a \in A} \sum_{\omega \in \Omega} (\bar{u} \circ F_a)(\omega)\, \sigma'(\omega)\, \alpha(a) = \sum_{m \in M} \bar{u}(m)\, \hat{F}_\alpha(\sigma)(m). \end{aligned}$$

This implies $U(\alpha,\sigma') = U(\alpha,\sigma)$ if $\hat{F}_\alpha(\sigma') = \hat{F}_\alpha(\sigma)$, that is, if $\sigma' \in \tau_\alpha(\sigma)$. ■

Identification correspondences with larger images exhibit a higher degree of partial identification. We now show that, ultimately, such degree depends on the underlying feedback functions. To this end, given any two feedback functions $f$ and $\bar{f}$, say that $f$ is *coarser* than $\bar{f}$ if it is $\bar{f}$-measurable, that is, $\bar{f}(z) = \bar{f}(z')$ implies $f(z) = f(z')$ for all $z, z' \in Z$.

For example, if $f = u$ (DM only observes his payoffs) and $\bar{f} = \mathrm{Id}_Z$ (DM observes the outcome, or terminal node of the decision tree), then $f$ is trivially coarser than $\bar{f}$. DMs with coarser feedback functions have worse information on outcomes. Clearly, constant feedbacks are the coarsest ones, while perfect feedbacks are the least coarse.

**Lemma 5** *If $f$ is coarser than $\bar{f}$, then, for all $\alpha \in \Delta(A)$,*

$$\bar{\tau}_\alpha(\sigma^*) \subseteq \tau_\alpha(\sigma^*) \qquad \forall \sigma \in \Delta(\Omega).$$

Coarser feedback functions thus determine, for each action, coarser identification correspondences: worse information translates into a higher degree of partial identification.

**Proof** There is a function $\varphi : \mathrm{Im}\,\bar{f} \to M$ such that $f = \varphi \circ \bar{f}$. Hence, $F = \varphi \circ \bar{F}$. Fix $\sigma^* \in \Delta(\Omega)$ and $\sigma \in \bar{\tau}_\alpha(\sigma^*)$, so that $\hat{\bar{F}}_\alpha(\sigma) = \hat{\bar{F}}_\alpha(\sigma^*)$. We have

$$\begin{aligned}
\sigma \in \tau_\alpha(\sigma^*) &\iff \hat{F}_\alpha(\sigma) = \hat{F}_\alpha(\sigma^*) \\
&\iff \sigma(F_\alpha = m) = \sigma^*(F_\alpha = m) \qquad \forall m \in M \\
&\iff \sigma\left(\varphi \circ \bar{F}_\alpha = m\right) = \sigma^*\left(\varphi \circ \bar{F}_\alpha = m\right) \qquad \forall m \in M \\
&\iff \sigma\left(\bar{F}_\alpha \in \varphi^{-1}(m)\right) = \sigma^*\left(\bar{F}_\alpha \in \varphi^{-1}(m)\right) \qquad \forall m \in M \\
&\iff \hat{\bar{F}}_\alpha(\sigma)\left(\varphi^{-1}(m)\right) = \hat{\bar{F}}_\alpha(\sigma^*)\left(\varphi^{-1}(m)\right) \qquad \forall m \in M
\end{aligned}$$

Since $\hat{\bar{F}}_\alpha(\sigma) = \hat{\bar{F}}_\alpha(\sigma^*)$, we conclude that $\sigma \in \tau_\alpha(\sigma^*)$. ■

We close with a remark on notation: in the sequel, we set $\Sigma(\alpha,\mu) = \hat{F}_\alpha^{-1}(\mu)$ and $\hat{\Sigma}(\alpha,\sigma) = \tau_\alpha(\sigma)$. By (5),

$$\hat{\Sigma}(\alpha,\sigma) = \Sigma(\alpha, \hat{F}(\alpha \times \sigma)). \tag{6}$$

## 2.5 Selfconfirming decisions

We begin with the general notion of selfconfirming decision for the smooth classical criterion (4). We focus on the case where DMs can only choose pure actions. Given an action distribution $\alpha$ in the population of DMs, a key issue for this notion is whether DMs have access to a kind of public database (e.g. provided by the media), or obtain only the observations generated when they choose their action $a$ when they are drawn (individual database). If $\sigma$ is the true model, $\hat{\Sigma}(\alpha,\sigma)$ is the set of models consistent with the message distribution in the public database, while $\hat{\Sigma}(a,\sigma)$ is the set of models consistent with individual database of a DM playing $a$. Though we will focus on individual databases, a similar analysis can be carried out for public ones (see the discussion in Section 7.3).

That said, here is our general notion of selfconfirming decision with individual databases. The goal is to describe distributions of actions/strategies that are stationary because each DM's beliefs are confirmed.

**Definition 6** *Suppose $\sigma^*$ is the true model. An action distribution $\alpha^*$ is a* smooth selfconfirming decision (SCD) *if, for each $a^* \in \operatorname{supp} \alpha^*$, there is a prior $p^* \in \Delta(\hat{\Sigma}(a^*, \sigma^*))$ such that*

$$V\left(a^*, p^*, \hat{\Sigma}(a^*, \sigma^*); \phi\right) \geq V\left(a, p^*, \hat{\Sigma}(a^*, \sigma^*); \phi\right) \qquad \forall a \in A \tag{7}$$

The "confirmed rationality" condition (7) can be written as

$$a^* \in \arg\max_{a \in A} \int_{\hat{\Sigma}(a^*, \sigma^*)} \phi(U(a, \sigma))\, dp^*(\sigma)$$

and ensures that $a^*$ is a best reply that takes into account the available evidence. Specifically, each action $a^*$ chosen by a positive fraction $\alpha^*(a^*)$ of DMs must be a best response within $A$ to the statistical distribution of messages $\hat{F}(a^* \times \sigma^*) \in \Delta(M)$ generated by selecting $a^*$ when the true model is $\sigma^*$. Since the available evidence depends on the chosen action $a^*$, also the belief "justifying" $a^*$ as a best response may depend on $a^*$. Since $A$ is finite, for any given $p^*$ the maximization problem has a solution.[21]

Extreme ambiguity attitudes determine two notions of selfconfirming decision. Specifically, an action distribution $\alpha^*$ is a:

(i) *Waldean SCD* if, for each $a^* \in \operatorname{supp} \alpha^*$,

$$a^* \in \arg\max_{a \in A} \min_{\sigma \in \hat{\Sigma}(a^*, \sigma^*)} U(a, \sigma);$$

(ii) *Bayesian SCD* if, for each $a^* \in \operatorname{supp} \alpha^*$, there is a prior $p^* \in \Delta(\hat{\Sigma}(a^*, \sigma^*))$ such that

$$a^* \in \arg\max_{a \in A} \int_{\hat{\Sigma}(a^*, \sigma^*)} U(a, \sigma)\, dp^*(\sigma).$$

Note that when the decision problem is identified under all actions – that is, $\hat{\Sigma}(a, \sigma) = \{\sigma\}$ for all $a \in A$ and all $\sigma \in \Delta(\Omega)$ – confirmed rationality require consistency with the true model and the smooth criterion reduces to a vNM expected utility with respect to the true model. The confirmed rationality condition becomes $a^* \in \arg\max_{a \in A} U(a, \sigma^*)$.

## 2.6 An urn example

Consider an urn that contains 90 balls: 30 are red ($R$), 30 are blue ($B$), and 30 are yellow ($Y$). In every period a DM is randomly selected from a population and is asked to bet 1 euro on the color of the ball that will be drawn from the urn in that period. The selected DMs just know that $R$, $B$, and $Y$ are the only possible colors. After each draw the DMs are told the result of their bets, that is, whether or not they won 1 euro.

[21] Note, since we are considering the case of individual databases and $\sigma^*$ is exogenously given, the equilibrium conditions only restrict the support of $\alpha^*$. But in the game theoretic extension to follow, $\sigma^*$ will be the "endogenous" distribution of strategies of a co-player who is also maximizing given confirmed beliefs and the equilibrium conditions will restrict also the probabilities of strategies in the support of $\alpha^*$.

Here the state space is $\Omega = \{R, B, Y\}$, the action space is $A = \{1_R, 1_B, 1_Y\}$, the set of terminal nodes of the decision tree is $Z = A \times \Omega$, hence $\zeta = \mathrm{Id}_Z$, and the message space is $M = \{0, 1\}$. Since $\zeta = \mathrm{Id}_Z$, $f = F : A \times \Omega \to \mathbb{R}$; in particular:

$$F(1_R, R) = F(1_B, B) = F(1_Y, Y) = 1$$

and

$$F(1_R, B) = F(1_R, Y) = F(1_B, R) = F(1_B, Y) = F(1_Y, R) = F(1_Y, B) = 0.$$

Suppose a DM keeps betting on red, that is, his action is $1_R$. As a result, he imperfectly observes the draws: he can only observe whether the color is red (message 1: he won) or not (message 0: he lost). Such DM's action prevents him from obtaining evidence on the frequency of $B$ and $Y$: given his betting he only gets either message 0 or 1. In particular,

$$\hat{F}_{1_R}(\sigma) = \{\mu \in \Delta(\{0,1\}) : \mu(1) = \sigma(R)\} \qquad \forall \sigma \in \Delta(\Omega).$$

Suppose one third of the draws yield message 1 and two third yield message 0, that is, $\mu \in \Delta(\{0,1\})$ given by

$$\mu(1) = \frac{1}{3} \quad , \quad \mu(0) = \frac{2}{3}$$

is the long run frequency distribution of messages that such DM receives. He thus learns that 1/3 of the balls are red, at most 2/3 are blue, at most 2/3 are yellow, which give rise to the set

$$\begin{aligned} \Sigma(1_R, \mu) &= \hat{F}_{1_R}^{-1}(\mu) = \left\{\sigma \in \Delta(R, B, Y) : \hat{F}(1_R \times \sigma) = \mu(1)\right\} \\ &= \left\{\sigma \in \Delta(R, B, Y) : \sigma(R) = \frac{1}{3}\right\} \end{aligned}$$

is the collection of all possible urn's compositions given their bets on $R$ and frequencies of messages received $\mu$. Identifiability is thus only partial.

Since the true model $\sigma^* \in \Delta(R, B, Y)$ is such that

$$\sigma^*(R) = \sigma^*(B) = \sigma^*(Y) = \frac{1}{3}$$

we can write:

$$\begin{aligned} \hat{\Sigma}(1_R, \sigma^*) &= \tau_{1_R}(\sigma^*) = \left\{\sigma \in \Delta(R, B, Y) : \hat{F}(1_R \times \sigma) = \hat{F}(1_R \times \sigma^*)\right\} \\ &= \{\sigma \in \Delta(R, B, Y) : \sigma(R) = \sigma^*(R)\} \\ &= \left\{\sigma \in \Delta(R, B, Y) : \sigma(R) = \frac{1}{3}\right\} \end{aligned}$$

The sets $\Sigma(1_R, \mu)$ and $\hat{\Sigma}(1_R, \sigma^*)$ are equal, despite capturing respectively a subjective and an objective perspective, because $\mu = F_{1_R}(\sigma^*)$.

Turning to selfconfirming decisions, set $u(0) = 0$ and $u(1) = 1$, so that payoffs are observable and $U(1_\omega, \sigma) = \sigma(\omega)$ for each $\omega \in \{R, B, Y\}$. Since

$$\min_{\sigma \in \hat{\Sigma}(1_R, \sigma^*)} U(1_R, \sigma) = \frac{1}{3} \quad \text{and} \quad \min_{\sigma \in \hat{\Sigma}(1_R, \sigma^*)} U(1_B, \sigma) = \min_{\sigma \in \hat{\Sigma}(1_R, \sigma^*)} U(1_Y, \sigma) = 0$$

action $1_R$, i.e., betting on red, is a strict Waldean SCD. A similar conclusion can be reached for smooth selfconfirming decisions under enough ambiguity aversion.

In a Bayesian setting, each prior $p \in \Delta\left(\hat{\Sigma}(1_R, \sigma^*)\right)$ induces a predictive $\bar{p} \in \hat{\Sigma}(1_R, \sigma^*)$ given by

$$\bar{p}(R) = \int_{\hat{\Sigma}(1_R,\sigma^*)} \sigma(R) dp(\sigma) = \frac{1}{3}$$

so that $U(1_R, \bar{p}) = 1/3$. Consider two cases:

(i) $\bar{p}(B) = \bar{p}(Y)$: then $\bar{p}(B) = \bar{p}(Y) = 1/3$, and so $U(1_B, \bar{p}) = U(1_Y, \bar{p}) = 1/3$. In this case $1_R$ is a Bayesian selfconfirming decision (though not strict, i.e., the $\arg\max$ is nonsingleton).

(ii) $\bar{p}(B) \neq \bar{p}(Y)$: wlog suppose $\bar{p}(B) > \bar{p}(Y)$, and so $\bar{p}(B) > 1/3$, which in turn implies $U(1_B, \bar{p}) > 1/3$. In this case $1_R$ is not a Bayesian selfconfirming decision.

# 3 Recurrent games

In this section we extend the previous analysis to general games, our main object of interest.

## 3.1 Games with feedback and ambiguity

We consider a finite extensive-form game with perfect recall and no chance moves played recurrently between agents drawn at random from large populations. Since we need not be explicit about all the details of the extensive form, we specify our notation only for some primitive and derived objects. The *rules of the game* directly or indirectly determine the following elements

- populations, or player roles $i \in I$,
- complete paths, or terminal histories $z \in Z$,
- pure strategies $s_i \in S_i$ for each $i \in I$,[22]
- a path function $\zeta : S \to Z$ specifying the the terminal history $\zeta(s)$ determined by each strategy profile $s = (s_i)_{i \in I} \in S = \times_{i \in I} S_i$,
- a consequence function $\gamma : Z \to C$ specifying the material consequences $c \in C$ of each terminal history $z \in Z$,
- a feedback function $f_i : Z \to M$ for each $i \in I$ specifying the message received *ex post* by an agent playing in role $i$ as a function of the terminal history $z$.

It may be useful to think of the above elements as what can be implemented in a laboratory experiment. In particular, the designer specifies the rules. Such rules include the ex post information feedback $f = (f_i)_{i \in I}$, which typically does not appear in the standard mathematical definition of game, and a function $\gamma = (\gamma_i)_{i \in I} : Z \to \mathbb{R}^I$ that gives the monetary payoff distributions induced by sequences of actions (thus, $C \subseteq \mathbb{R}^I$ in this example).

[22] Of course, $S_i$ is determined by the game tree and the information structure for $i$.

On the other hand, the designer cannot control subjects' preferences, notably absent from the above list.

But, as analysts, we need to specify preferences over objective and subjective lotteries of consequences. We assume for simplicity that populations are homogeneous, that is, all agents in the same population $i \in I$ have the same preferences:

- $v_i : C \to \mathbb{R}$ is a vNM utility function capturing $i$'s attitudes toward risk,[23]
- $\phi_i : [\min_{c\in C} u_i(c), \max_{c\in C} u_i(c)] \to \mathbb{R}$ is a strictly increasing weighting function capturing $i$'s attitudes toward ambiguity as in KMM.

Like the feedback functions $f$, the weighting functions $\phi = (\phi_i)_{i\in I}$ do not appear in the standard theory, as they are implicitly assumed to be linear. The standard definition of game is a structure[24]

$$\Gamma = (I, ..., \gamma, (v_i)_{i\in I}),$$

where the dots refer to the details of the game tree and information structure. We keep the parameters specified by $\Gamma$ fixed throughout our analysis, while we look at the effects of changing $f$ or, more importantly, $\phi$.

A *game with feedback* is a pair $(\Gamma, f)$. We say that such a game features *observable payoffs* when each payoff function $u_i = v_i \circ \gamma : Z \to \mathbb{R}$ is $f_i$-measurable, that is, $f_i(z) = f_i(z')$ implies $u_i(z) = u_i(z')$ for all $z, z' \in Z$. An important special case is *perfect feedback*, where each $i$ directly observes the path, that is, $M = Z$ and $f_i = \mathrm{Id}_Z$ for each $i$.[25] To anticipate, some notions of selfconfirming equilibrium, including the standard one which assumes Bayesian agents, refer to a game with feedback. But our main equilibrium concept refers to a *game with feedback and ambiguity attitudes*, that is, a triple $(\Gamma, f, \phi)$.

As a matter of interpretation, we have to ascribe some knowledge of $(\Gamma, f, \phi)$ to the agents. In order to make sense of the following analysis, we informally assume that each agent in population $i$ knows the extensive form, the feedback function $f_i$, the random matching structure, and – of course – the utility function $v_i$ and weighting function $\phi_i$. No additional knowledge, mutual knowledge of common knowledge of $(\Gamma, f, \phi)$ is required until further notice (see our discussion of rationalizable selfconfirming equilibrium in Section 7.2). The assumption of no chance move is made for simplicity, without substantial loss of generality. Chance can be modeled in this framework as a player with a constant utility function, assuming that agents have no knowledge of the objective probabilities of chance moves. The analysis can be easily adapted to incorporate complete or partial knowledge of such probabilities.

The (*strategic-form*) *message function* $F_i : S \to M$ is given by $F_i = f_i \circ \zeta$. In turn, each message function $F_i : S \to M$ induces a message distribution function $\hat{F}_i : \Delta(S) \to \Delta(M)$,

[23] For example, if $\gamma = (\gamma_i) : Z \to \mathbb{R}$ and $i$ is selfish, $u_i(\gamma(z)) = v_i(\gamma_i(z))$, where $v_i$ is concave if $i$ is risk averse.

[24] Often the consequence function does not appear explicitly. We introduce it here for conceptual clarity.

[25] Perfect feedback was assumed by Fudenberg and Levine (1993a) and Fudenberg and Kreps (1995). It is also reasonable to assume that $f_i$ reflects perfect recall, that is, if the sequence of information sets and actions of $i$ determined by path $z'$ is different from the sequence determined by path $z''$, then $f_i(z') \neq f_i(z'')$. This, however, does not play an explicit role in our analysis because we assume that an agent takes into account the strategy he is playing when he considers the possible strategy distributions of the opponents consistent with the evidence he observes.

where $\hat{F}_i(\sigma_i \times \sigma_{-i})(m)$ is the probability that $i$ observes message $m$ given the strategy distribution $\sigma_i$ of his role and the strategy distribution $\sigma_{-i}$ of his opponents' roles.

If $i$ plays the pure strategy $s_i$ and observes the long-run frequency distribution of messages $\mu \in \Delta(M)$, then $i$ can compute the set of (product) strategy distributions of the opponents that may have generated $\mu$ given $s_i$:

$$\Sigma_{-i}(s_i, \mu) = \left\{ \sigma_{-i} \in \bigotimes_{j \neq i} \Delta(S_j) : \hat{F}_i(s_i \times \sigma_{-i}) = \mu \right\}.$$

If $\sigma_{-i} = \times_{j \neq i} \sigma_j$ is the strategy distribution of his opponents' roles, the long-run frequency distribution of messages observed by $i$ is the one induced by the objective distribution $s_i \times \sigma_{-i}$, that is, $\mu = \hat{F}_i(s_i \times \sigma_{-i})$. The set of possible distributions from $i$'s perspective is thus

$$\hat{\Sigma}_{-i}(s_i, \sigma_{-i}) = \Sigma_{-i}\left(s_i, \hat{F}_i(s_i \times \sigma_{-i})\right),$$

which is the game theoretic version of (6). Note that $\Sigma_{-i}(s_i, \mu)$ is a compact subset of $\bigotimes_{j \neq i} \Delta(S_j)$. It is convex in two-person games, and "coordinate-wise convex" in games with perfect feedback.[26] The same holds for $\hat{\Sigma}_{-i}(s_i, \sigma_{-i})$, which is also nonempty, as it contains $\sigma_{-i}$.

We obtain the decision problems analyzed above as a special case when $I = \{DM, N\}$ and $u_N$ is constant. In particular, the decision criteria previously introduced are easily extended to this game theoretic setting. Here (1) becomes

$$U_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} u_i(\zeta(s_i, s_{-i}))\sigma_{-i}(s_{-i}),$$

with $u_i = v_i \circ \gamma$, so that the smooth criterion (4) becomes

$$V_i(s_i, p_i, \Sigma_{-i}; \phi_i) = \phi_i^{-1}\left(\int_{\Sigma_{-i}} \phi_i(U(s_i, \sigma_{-i}))\, dp_i(\sigma_{-i})\right)$$

where $p_i \in \Delta(\Sigma_{-i})$ is a prior of $i$ on his opponents' possible strategy distributions. The Waldean and Bayesian criteria are, respectively,

$$V_i(s_i, \Sigma_{-i}) = \min_{\sigma_{-i} \in \Sigma_{-i}} U_i(s_i, \sigma_{-i})$$

and

$$V_i(s_i, p_i, \Sigma_{-i}) = \int_{\Sigma_{-i}} U_i(s_i, \sigma_{-i})\, dp(\sigma_{-i}).$$

[26] More generally, in games with "ex post observable deviators," i.e., games where $F_i^{-1}(m)$ is a cross-product for each $i$ and $m$. On "observable deviators" see Fudenberg and Levine (1993a) and Battigalli (1997).

### 3.2 Selfconfirming equilibrium

Next we give a general definition of selfconfirming equilibrium for the smooth criterion that restricts agents to choose pure strategies, so that "mixed" strategies arise only as distributions of pure strategies within populations of agents.[27]

**Definition 7** *A profile of strategy distributions $\sigma^* = (\sigma_i^*)_{i\in I}$ is a* smooth selfconfirming equilibrium (SSCE) *of a game with feedback and ambiguity attitudes $(\Gamma, f, \phi)$ if, for each $i \in I$ and each $s_i^* \in \operatorname{supp}\sigma_i^*$, there is a prior $p_i^* \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ such that*

$$V_i\left(s_i^*, p_i^*, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*); \phi_i\right) \geq V_i\left(s_i, p_i^*, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*); \phi_i\right) \tag{8}$$

*for each $s_i \in S_i$.*

This "confirmed rationality" condition extends condition (7) to general games and can be written as

$$s_i^* \in \arg\max_{s_i\in S_i} \int_{\hat{\Sigma}_{-i}(s_i^*,\sigma_{-i}^*)} \phi_i\left(U_i\left(s_i, \sigma_{-i}\right)\right) dp_i^*(\sigma_{-i}).$$

It requires that every pure strategy $s_i^*$ that a positive fraction $\sigma_i^*(s_i)$ of agents keep playing must be a best response within $S_i$ to the "evidence", that is, the statistical distribution of messages $\hat{F}_i(s_i, \sigma_{-i}^*) \in \Delta(M)$ generated by playing $s_i^*$ against the strategy distribution $\sigma_{-i}^*$.

A profile $\sigma^* = (\sigma_i^*)_{i\in I}$ is a:

(i) *Waldean selfconfirming equilibrium (WSCE)* if, for each $i \in I$ and each $s_i^* \in \operatorname{supp}\sigma_i^*$,

$$s_i^* \in \arg\max_{s_i\in S_i} \min_{\sigma_{-i}\in\hat{\Sigma}_{-i}(s_i^*,\sigma_{-i}^*)} U_i\left(s_i, \sigma_{-i}\right); \tag{9}$$

(ii) *Bayesian selfconfirming equilibrium (BSCE)* if, for each $i \in I$ and each $s_i^* \in \operatorname{supp}\sigma_i^*$, there is a prior $p_i^* \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ such that

$$s_i^* \in \arg\max_{s_i\in S_i} \int_{\hat{\Sigma}_{-i}(s_i^*,\sigma_{-i}^*)} U_i\left(s_i, \sigma_{-i}\right) dp_i^*\left(\sigma_{-i}\right); \tag{10}$$

(iii) *symmetric selfconfirming equilibrium* if $\sigma^*$ is a pure strategy profile, that is, all agents in the same role play the same pure strategy.

The definition of Bayesian selfconfirming equilibrium subsumes the definition of conjectural equilibrium due to Battigalli (1987) (see also Battigalli and Guaitoli, 1997) and the definition of selfconfirming equilibrium of Fudenberg and Levine (1993a) as special cases. Battigalli (1987) allows for general feedback functions $f_i$, but considers only symmetric equilibria and assumes independent beliefs.[28] Fudenberg and Levine (1993a) implicitly assume that $f_i$ is the identity for each $i$ (players perfectly observe the whole path of play).

[27] At the interpretive level, we are not assuming that agents are prevented from using randomization devices: it may be the case that agents in population $i$ have a set $\hat{S}_i \subset S_i$ of "truly pure" strategies and that $S_i$ includes a finite set of choices that are realization equivalent to randomizations over $\hat{S}_i$. Of course the definition of $F_i$ has to be adapted accordingly, as $F_i(s_i, s_{-i})$ is a random message when $s_i$ is a randomization device.

[28] Battigalli (1987) also allows for randomization. Equilibria where agents choose randomized strategies are discussed in the next section.

Like these earlier notions of SCE, our more general notion is motivated by a partial identification problem: the mapping from strategy distributions to the distributions of observations available to an agent is not one to one. In fact, if for each agent $i$ identification is full – i.e., $\hat{\Sigma}_{-i}(s_i, \sigma_{-i}) = \{\sigma_{-i}\}$ for all $s_i$ and all $\sigma_{-i}$ – condition (8) is easily seen to reduce to the standard Nash equilibrium condition $U_i\left(s_i^*, \sigma_{-i}^*\right) \geq U_i\left(s_i, \sigma_{-i}^*\right)$. In other words, if none of the agents features a partial identification problem, we are back to the Nash equilibrium notion (in its mass action interpretation).

Finally, note that ambiguity aversion may give rise to dynamic inconsistency issues: as the play unfolds, an agent may have incentives to deviate from an ex ante optimal strategy. As we mentioned, we avoid this problem by assuming that $i$ just commits to a strategy that is then automatically implemented. We already clarified that it is important for ambiguity averse agents whether randomization is explicitly allowed or instead mixed strategies $\sigma_i$ arise as pure strategy distributions in the population of agents who play in role $i$: we will see that the definition of SCE where agents cannot randomize is not a special case of the definition of SCE where agents can choose any randomization they like. In Sections 6 and 7 we will say more on this.

# 4 Comparative statics and relationships

Looking at games with feedback and ambiguity attitudes, we can carry out some key comparative static analysis in information and ambiguity. From this we also derive some results about relationships between equilibrium concepts. To this end, observe that:

(i) a feedback profile $f$ is *coarser* than $\bar{f}$ if, for each $i$ , $f_i$ is coarser than $\bar{f}_i$;

(ii) an ambiguity profile $\phi$ is *more ambiguity averse* than $\bar{\phi}$ if, for each $i$, $\phi_i$ is *more ambiguity averse* than $\bar{\phi}_i$.[29]

A coarser feedback characterizes agents with less private ex post information, while higher ambiguity aversion characterizes agents with a higher dislike for model uncertainty. Clearly, any feedback profile is coarser than the perfect feedback profile $f = (\mathrm{Id}_Z, ..., \mathrm{Id}_Z)$, while any ambiguity averse profile $\phi$ is more ambiguity averse than the ambiguity neutral profile $\phi = (\mathrm{Id}_{\mathbb{R}}, ..., \mathrm{Id}_{\mathbb{R}})$.

Accordingly, we say that:

(i) game $(\Gamma, f, \phi)$ has *coarser feedback* than $\left(\Gamma, \bar{f}, \phi\right)$ if $f$ is coarser than $\bar{f}$;

(ii) game $(\Gamma, f, \phi)$ is *more ambiguity averse* than $\left(\Gamma, f, \bar{\phi}\right)$ if $\phi$ is more ambiguity averse than $\bar{\phi}$.

Note that the comparison of ambiguity attitudes does not require that the profiles themselves be ambiguity averse. It only matters that one profile be comparatively more ambiguity averse than the other (something that can happen even if both are ambiguity loving).

[29] That is, there is a concave and strictly increasing function $\varphi_i : \mathrm{Im}\,\phi_i \to \mathbb{R}$ such that $\bar{\phi}_i = \varphi_i \circ \phi_i$. On this comparative notion see KMM.

We can now turn to the comparative static analysis. We begin by studying how equilibria are affected by changes in information feedback.[30]

**Proposition 8** *Suppose* $(\Gamma, f, \phi)$ *has coarser feedback than* $\left(\Gamma, \bar{f}, \phi\right)$*. Then, the SSCEs of* $\left(\Gamma, \bar{f}, \phi\right)$ *are also SSCEs of* $(\Gamma, f, \phi)$*, while the WSCEs of* $\left(\Gamma, \bar{f}\right)$ *are WSCEs of* $(\Gamma, f)$*.*

**Proof** Let $\sigma^*$ be a SSCE for less coarse game $\left(\Gamma, \bar{f}, \phi\right)$. By Lemma 5, $\hat{\bar{\Sigma}}_{-i}(s_i^*, \sigma_{-i}^*) \subseteq \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$ for each $i$ and $s_i^* \in \operatorname{supp} \sigma_i^*$, which easily implies that $\sigma^*$ be a SSCE for $(\Gamma, f, \phi)$. A similar argument holds for WSCE. ■

Worse feedback thus enlarges the set of SSCEs and WSCEs. Next we show that the same is true under higher ambiguity aversion. It is the paper main result, which relies on the status quo bias of Lemma 4.

**Theorem 9** *Suppose* $(\Gamma, f, \phi)$ *is more ambiguity averse than* $\left(\Gamma, f, \bar{\phi}\right)$*. If payoffs are observable, then the SSCEs of* $\left(\Gamma, f, \bar{\phi}\right)$ *are also SSCEs of* $(\Gamma, f, \phi)$*. Similarly, the SSCEs of any game* $(\Gamma, f, \phi)$ *are also WSCEs.*

**Proof** By definition, there exists a profile $(\varphi_i)_{i\in I}$ of strictly increasing and concave functions such that $\phi_i = \varphi_i \circ \bar{\phi}_i$ for each $i$. Let $\sigma^*$ be a SSCE of $\left(\Gamma, f, \bar{\phi}\right)$. Fix $i \in I$, and pick $p_i^* \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ and $s_i^* \in \operatorname{supp} \sigma_i^*$ such that $s_i^* \in \arg\max_{s_i \in S_i} V_i(s_i, p_i^*; \bar{\phi}_i)$ (we omit for simplicity the set of possible models from the smooth ambiguity value function). We want to show that $s_i^* \in \arg\max_{s_i \in S_i} V_i(s_i, p_i^*; \phi_i)$, which implies the first claim. Since payoffs are observable, by Lemma 4 $U_i(s_i^*, \sigma_{-i}) = U_{-i}(s_i^*, \sigma_{-i}^*)$ for each $\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$. Thus

$$V_i(s_i^*, p_i^*; \phi_i) = U_i(s_i^*, \sigma_{-i}^*) = V_i(s_i^*, p_i^*; \bar{\phi}_i). \tag{11}$$

Next observe that, for any $s_i \in S_i$,

$$\begin{aligned}
V_i(s_i, p_i^*; \bar{\phi}_i) &= \bar{\phi}_i^{-1} \left( \int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \bar{\phi}_i(U_i(s_i, \sigma_{-i})) dp_i^*(\sigma_{-i}) \right) \\
&= (\bar{\phi}_i^{-1} \circ \varphi_i^{-1}) \circ \varphi_i \left( \int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \bar{\phi}_i(U_i(s_i, \sigma_{-i})) dp_i^*(\sigma_{-i}) \right) \\
&\geq (\bar{\phi}_i^{-1} \circ \varphi_i^{-1}) \left( \int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} (\varphi_i \circ \bar{\phi}_i)(U_i(s_i, \sigma_{-i})) dp_i^*(\sigma_{-i}) \right) \\
&= \phi_i^{-1} \left( \int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \phi_i(U_i(s_i, \sigma_{-i})) dp_i^*(\sigma_{-i}) \right) = V_i(s_i, p_i^*; \phi_i)
\end{aligned}$$

where we used Jensen's inequality and $\phi_i = \varphi_i \circ \bar{\phi}_i$. Hence,

$$V_i(s_i, p_i^*; \phi_i) \leq V_i(s_i, p_i^*; \bar{\phi}_i) \leq V_i(s_i^*, p_i^*; \bar{\phi}_i) = V_i(s_i^*, p_i^*; \phi_i)$$

[30] Similar results are part of the folkore on SCE (see, e.g., Fudenberg and Kamada, 2011). We do not provide such proposition because of its originality, but rather because it sharpens the reader's understanding of the framework and concepts.

for each $s_i \in S_i$, which shows that $s_i^* \in \arg\max_{s_i \in S_i} V_i(s_i, p_i^*; \phi_i)$.

We now prove that SSCEs are WSCEs. Let $\sigma^*$ be a SSCE of a game $(\Gamma, f, \phi)$. Fix $i \in I$, and pick $p_i^* \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ and $s_i^* \in \operatorname{supp} \sigma_i^*$ such that $s_i^* \in \arg\max_{s_i \in S_i} V_i(s_i, p_i^*; \phi_i)$. We show that $s_i^* \in \arg\max_{s_i \in S_i} V_i(s_i, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ where

$$V_i(s_i, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)) = \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U\left(s_i, \sigma_{-i}\right)$$

Since payoffs are observable, by (11) it holds

$$V_i(s_i^*, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)) = U_i(s_i^*, \sigma_{-i}^*) = V_i(s_i^*, p_i^*; \phi_i) \geq V_i(s_i, p_i^*; \phi_i) \tag{12}$$

for each $s_i \in S_i$. Next observe that for, each $\sigma'_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$ and each $s_i \in S_i$,

$$\begin{aligned} & U_i(s_i, \sigma'_{-i}) \geq \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U\left(s_i, \sigma_{-i}\right) \\ \Longrightarrow \quad & \phi_i\left(U_i(s_i, \sigma'_{-i})\right) \geq \phi_i\left(\min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U\left(s_i, \sigma_{-i}\right)\right). \end{aligned}$$

This implies that, for each $\sigma'_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$ and each $s_i \in S_i$,

$$\int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \phi_i\left(U_i(s_i, \sigma'_{-i})\right) dp_i^*(\sigma'_{-i}) \geq \int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \phi_i\left(\min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U\left(s_i, \sigma_{-i}\right)\right) dp_i^*(\sigma'_{-i})$$

which in turn implies

$$\begin{aligned} V_i(s_i, p_i; \phi_i) &= \phi_i^{-1}\left(\int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \phi_i\left(U_i(s_i, \sigma'_{-i})\right) dp_i^*(\sigma'_{-i})\right) \\ &\geq \phi_i^{-1}\left(\int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \phi_i\left(\min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U\left(s_i, \sigma_{-i}\right)\right) dp_i^*(\sigma'_{-i})\right) \\ &= V_i(s_i, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)). \end{aligned}$$

This latter inequality paired with (12) delivers that

$$V_i(s_i^*, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)) \geq V_i(s_i, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)) \qquad \forall s_i \in S_i$$

proving the statement. ■

In sum, the set of SSCEs increases as either ex post private information becomes coarser or ambiguity aversion increases (or both). In particular, if we fix a game with feedback $(\Gamma, f)$, Theorem 9 implies that under observable payoffs:

(i) the set of BSCEs of $(\Gamma, f)$ is contained in the set of SSCEs of every $(\Gamma, f, \phi)$ with ambiguity averse players;

(ii) the set of SSCEs of every $(\Gamma, f, \phi)$ is contained in the set of WSCEs of $(\Gamma, f)$.

In other words, under observable payoffs and ambiguity aversion, it holds

$$BSCE \subseteq SSCE \subseteq WSCE. \tag{13}$$

The degree of ambiguity aversion determines the size of the set of selfconfirming equilibria, with the sets of Bayesian and Waldean selfconfirming equilibria being, respectively, the smallest and largest one.

**Example 10** *In the next section the game depicted in Figure 3 of the Introduction will be used to show that the inclusions in (13) can be strict. Here we note that it also shows that the assumption of observable payoffs is crucial. For, suppose that player* 1 *cannot observe his payoff ex post (he only remembers his action). For example, his payoff function may just reflect a psychological preference for increasing the utility of player* 2*. Then, even if he plays either* $L$ *or* $R$ *for a long time he gets no feedback and the maxmin value of both actions is zero. Thus,* $A$ *would be the only WSCE strategy of* 1 *under this violation of the observable payoff assumption since* $\hat{\Sigma}_{-1}(s_1, \sigma_2) = \Delta(\{l, r\})$ *for all* $(s_1, \sigma_2)$*. But* $A$ *is strictly dominated, and so it cannot be a BSCE strategy; thus,* $BSCE \cap WSCE = \emptyset$ *in this case. This also shows why, under observable payoffs, the inclusions above may be strict: it can be checked that, for any feedback function,* $A$ *is a WSCE action, while whether* $A$ *is a SSCE action or not depends on the concavity of* $\phi_1$*. We will come back to this game again when we discuss randomization (see Example 20).* ▲

As well known, every Nash equilibrium $\sigma^*$ of game $\Gamma$ is a BSCE of $(\Gamma, f)$.

**Lemma 11** *If* $\sigma^*$ *is a Nash equilibrium of* $\Gamma$*, it is a BSCE of any game* $(\Gamma, f)$ *with feedback.*

For the sake of completeness we give a proof in our setup.

**Proof** Pick any $i$ and pure strategy $s_i^* \in \operatorname{supp} \sigma_i^*$. Then $U_i\left(s_i^*, \sigma_{-i}^*\right) \geq U_i\left(s_i, \sigma_{-i}^*\right)$ for each $s_i \in S_i$. Given any feedback function $f$ it holds $\sigma_{-i}^* \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$. Hence, $\delta_{\sigma_{-i}^*} \in \Delta\left(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)\right)$. Since $V_i\left(s_i, \delta_{\sigma_{-i}^*}, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)\right) = U_i\left(s_i, \sigma_i^*\right)$ for each $s_i \in S_i$, it follows that $\sigma^*$ is a BSCE of $(\Gamma, f)$. ■

Since the set $NE$ of Nash equilibria $\sigma^*$ is nonempty, we can thus enrich (13) as follows:

$$\emptyset \neq NE \subseteq BSCE \subseteq SSCE \subseteq WSCE.$$

This shows, under observable payoffs and ambiguity aversion, that every game $(\Gamma, f, \phi)$ has some SSCE and that Nash equilibria are not only BSCE, but more generally SSCE.

In two-person games, one can show directly, without assumptions about the feedback functions, that a BSCE is also a SSCE: the proof relies on the convexity of $\hat{\Sigma}_{-i}(s_i, \sigma_{-i})$ when $|I| = 2$ that allows to go from a justifying belief $p_i \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ to its predictive $\bar{p}_i \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$ to a new justifying belief $\delta_{\bar{p}_i} \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ inducing no subjective uncertainty.[31] This argument does not work with $n$-person games when $n \geq 3$.

[31] Given the previous example, one might conclude that there is a lack of upper-hemicontinuity of the equilibrium correspondence as the concavity of some $\phi_i$ goes to infinite. Such inference would be unjustified: the Waldean criterion is a limit of the smooth ambiguity criterion only when $p_i$ has full support in the set of possible models, but $p_i$ may have a smaller support.

**Example 12** *Consider the following three-person, simultaneous move game where player 1 chooses the row, player 2 the column and player 3 the matrix. Assume that player 3 has trivial feedback: he only remembers his action, hence* $\hat{\Sigma}_{-3}(s_3, \sigma_{1,2}) = \Delta(S_1) \bigotimes \Delta(S_2)$*. In contrast, players 1 and 2 have perfect feedback:*

| $l$ | $L$ | $R$ |
|---|---|---|
| $T$ | $1,1,2$ | $0,0,0$ |
| $B$ | $0,0,0$ | $1,1,0$ |

| $r$ | $L$ | $R$ |
|---|---|---|
| $T$ | $1,1,1$ | $0,0,1$ |
| $B$ | $0,0,1$ | $1,1,1$ |

*In equilibrium* $\sigma_1^*(T) = \sigma_1^*(L) \in \{0, 1, 1/2\}$*. Suppose that player 3 is certain that players 1 and 2 achieve perfect coordination, but he feels completely ignorant as to how. By symmetry, his belief is thus*

$$p_3 = \frac{1}{2}\delta_T \times \delta_L + \frac{1}{2}\delta_B \times \delta_R.$$

*If he is a Bayesian (linear* $\phi_3$*), then both* $l$ *and* $r$ *are best replies. But, if he is strictly uncertainty averse (strictly concave* $\phi_3$*), then only* $r$ *is a best reply. The predictive probability*

$$\frac{1}{2}(T, L) + \frac{1}{2}(B, R)$$

*is a correlated distribution in* $\Delta(S)$*, hence it cannot belong to* $\hat{\Sigma}_{-3}(l, \sigma^*_{-3})$*, and*

$$\delta_{\frac{1}{2}(T,L)+\frac{1}{2}(B,R)} \notin \Delta(\hat{\Sigma}_{-3}(l, \sigma^*_{-3})).$$

*Note, however, that we can support* $l$ *as a SSCE choice using different beliefs: this is not an example where some BSCE is not a SSCE.* ▲

Assuming perfect information and perfect feedback, we can prove a partial converse to the above inclusions and hence an equivalence result.

**Proposition 13** *In games with perfect information and perfect feedback, every symmetric WSCE is also a symmetric SSCE and BSCE where each player's strategy is justified by a confirmed deterministic belief.*

**Proof** As in Chapter 6 of Osborne and Rubinstein (1994), nodes of the game tree are histories (sequences of action profiles) $h$, with $\prec$ denoting the strict precedence "prefix of" relation, $H_i$ is the set of histories where $i$ moves, $H_{-i}$ is the complement of $H_i$ within the set of non-terminal histories, the action selected by strategy $s_i$ at $h \in H_i$ is denoted by $s_i(h)$.

We show that, if $s^*$ is a symmetric WSCE, then for each $i$ there is pure strategy profile $\bar{s}_{-i} \in \hat{\Sigma}_{-i}(s_i^*, s_{-i}^*)$ such that $s_i^* \in \arg\max_{s_i \in S_i} U_i(s_i, \bar{s}_{-i})$; this implies the claim. Let $s^* = (s_i^*)_{i\in I}$ be a symmetric WSCE. By perfect feedback, for each $i$, $\hat{\Sigma}_{-i}(s^*)$ is the convex hull of set of pure strategy profiles

$$\hat{S}_{-i}(s^*) := \{s_{-i} \in S_{-i} : \forall h \in H_{-i}, h \prec \zeta(s^*) \Rightarrow s_{-i}(h) = s^*_{-i}(h)\}.$$

This is the set of strategies of coalition $-i$ in a modified zero-sum game between $i$ and $-i$ where $-i$ has only one feasible action, $s^*_{-i}(h)$, at each $h \in H_{-i}$ preceding $\zeta(s^*)$, and the payoff function of $i$ is the restriction of $u_i$ on the smaller set of terminal histories thus obtained.

Like the original game, this auxiliary zero-sum game has perfect information, hence it has a saddle point in pure strategies. Taking into account that $s^*$ is a WSCE:[32]

$$s_i^* \in \arg\max_{s_i \in S_i} \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s^*)} U_i(s_i, \sigma_{-i}) = \arg\max_{s_i \in S_i} \min_{s_{-i} \in \hat{S}_{-i}(s^*)} U_i(s_i, s_{-i}).$$

Any strategy profile

$$\bar{s}_{-i} \in \arg\min_{s_{-i} \in \hat{S}_{-i}(s^*)} U_i(s_i^*, s_{-i}))$$

is a confirmed degenerate belief justifying $s_i^*$. ■

Since perfect feedback implies observable payoffs, Theorem 9 and Proposition 13 imply that under perfect information and perfect feedback[33]

$$symBSCE = symSSCE = symWSCE.$$

Note that the result about symSSCE holds with either two players or $n$ ambiguity averse players.

Games with perfect information necessarily feature observable deviators.[34] In games with observable deviators and perfect feedback, every symmetric BSCE has the same outcome as some mixed Nash equilibrium.[35] Therefore, in games with perfect information and perfect feedback the set terminal histories induced (with probability one) by some mixed Nash equilibrium coincides with the set $\zeta(symWSCE)$ of terminal histories induced by a symmetric WSCE. To summarize:

**Proposition 14** *In games with perfect information and perfect feedback the set of deterministic paths induced by some mixed Nash equilibrium is $\zeta(symWSCE) = \zeta(symBSCE)$; if, in addition, there are either two players or $n$ ambiguity averse players, then*

$$\zeta(symBSCE) = \zeta(symSSCE) = \zeta\left(symWSCE\right).$$

# 5 A common interest example

Consider the common interest game in Figure 3 of the Introduction, whose normal form is

| $1\diagdown 2$ | $l$ | $r$ |
|---|---|---|
| $A$ | $x, x$ | $x, x$ |
| $L$ | $3, 3$ | $0, 0$ |
| $R$ | $0, 0$ | $3, 3$ |

This game shows that the inclusions in (13) can be strict. We first show that, as argued in the Introduction, a key feature of this game is that strategy $A$ cannot have positive probability in any BSCE, but it is instead played in some WSCE.

[32] It does not matter how a strategy $s_i$ is specified at histories outside the modified game.
[33] The prefix *sym* denotes symmetric selfconfirming equilibria.
[34] For every path $z$ and every information set $\mathbf{h}$ reached with a unilateral deviation from $z$, the player moving at $\mathbf{h}$ can identify the deviator. See Fudenberg and Levine (1993a) and Battigalli (1997).
[35] A correct proof of a slightly more general result can be found in Kamada (2010).

**Lemma 15** *(i) In any BSCE $\sigma^*$, it holds $\sigma_1^*(A) = 0$.*

*(ii) All pairs $\{(A, \sigma_2^*) : \sigma_2^* \in \Delta(\{l, r\})\}$ are WSCEs.*

**Proof** (i) Given any $\sigma_2 \in \Delta(\{l, r\})$, we have

$$U_1(A, \sigma_2) = x \quad ; \quad U_1(L, \sigma_2) = 3\sigma_2(l) \quad ; \quad U_1(R, \sigma_2) = 3\sigma_2(r)$$

while given $\sigma_1 \in \Delta(\{A, L, R\})$, we have

$$U_2(l, \sigma_1) = x\sigma_1(A) + 3\sigma_1(L) \quad ; \quad U_2(r, \sigma_1) = x\sigma_1(A) + 3\sigma_1(R)$$

Hence, given any set $\Sigma \subseteq \Delta(\{l, r\})$ and any prior $p_1 \in \Delta(\Sigma)$ of the player in role 1, it holds

$$\begin{aligned} V_1(A, p_1, \Sigma) &= U_1(A, \bar{p}_1) = x \quad ; \quad V_1(L, p_1, \Sigma) = U_1(L, \bar{p}_1) = 3\bar{p}_1(l) \\ V_1(R, p_1, \Sigma) &= U_1(R, \bar{p}_1) = 3\bar{p}_1(r) \end{aligned}$$

Since

$$\max\{3\bar{p}_1(l), 3\bar{p}_1(r)\} = 3\max\{\bar{p}_1(l), 1 - \bar{p}_1(l)\} \geq 3/2 > x,$$

it follows that $V_1(A, p_1^*, \Sigma) < \max\{V_1(L, p_1, \Sigma), V_1(R, p_1, \Sigma)\}$, and so $A$ is never a best reply.

(ii) Given any $\sigma_2^* \in \Delta(\{l, r\})$, suppose players in role 1 keep playing $A$. It is easy to see that this implies that $\hat{\Sigma}_2(A, \sigma_2^*) = \Delta(\{l, r\})$, and so

$$\begin{aligned} V_1\left(A, \hat{\Sigma}_2(A, \sigma_2^*)\right) &= \min_{\sigma_2 \in \Delta(\{l,r\})} U_1(A, \sigma_2) = x \\ V_1\left(L, \hat{\Sigma}_2(A, \sigma_2^*)\right) &= \min_{\sigma_2 \in \Delta(\{l,r\})} U_1(L, \sigma_2) = \min_{\sigma_2 \in \Delta(\{l,r\})} 3\sigma_2(l) = 0 \\ V_1\left(R, \hat{\Sigma}_2(A, \sigma_2^*)\right) &= \min_{\sigma_2 \in \Delta(\{l,r\})} U_1(R, \sigma_2) = \min_{\sigma_2 \in \Delta(\{l,r\})} 3\sigma_2(r) = 0 \end{aligned}$$

Hence, $A \in \arg\max_{s_1 \in S_1} V_1\left(s_1, \hat{\Sigma}_2(A, \sigma_2^*)\right)$. On the other hand, suppose a proportion $\sigma_2^*(l)$ of players in role 2 keeps playing $l$ and a proportion $\sigma_2^*(r)$ keeps playing $r$. Then, $\hat{\Sigma}_1(l, A) = \hat{\Sigma}_1(r, A) = \{\delta_A\}$, and so

$$V_2\left(l, \hat{\Sigma}_1(\sigma_2^*, A)\right) = U_2(l, \delta_A) = x \quad ; \quad V_2\left(r, \hat{\Sigma}_1(\sigma_2^*, A)\right) = U_2(r, \delta_A) = x$$

This proves that $\{l, r\} = \arg\max_{s_2 \in S_2} V_2\left(s_2, \hat{\Sigma}_1(\sigma_2^*, A)\right)$. ■

After having established this noteworthy feature of strategy $A$, we characterize the set of all WSCEs.

**Proposition 16** *$(\sigma_1^*, \sigma_2^*)$ is a WSCE if and only if either*

$$\sigma_1^*(A) = 1 \quad \textit{and} \quad \sigma_2^*(r) \in [0, 1]$$

*or*

$$\sigma_1^*(A) < 1 \quad \textit{and} \quad \frac{\sigma_1^*(R)}{1 - \sigma_1^*(A)} = \sigma_2^*(r) \in \left\{0, \frac{1}{2}, 1\right\}$$

**Proof** Given any $\sigma_2^* \in \Delta(\{l,r\})$, if players in role 1 keep playing $L$ the frequency of plays $l$ (they won 3) and $r$ (they won 0) that they observe are, respectively, $\sigma_2^*(l)$ and $\sigma_2^*(r)$. Hence $\hat{\Sigma}_2(L,\sigma_2^*) = \{\sigma_2^*\}$. Similarly, $\hat{\Sigma}_2(R,\sigma_2^*) = \{\sigma_2^*\}$. Finally, it is easy to see that $\hat{\Sigma}_2(A,\sigma_2^*) = \Delta(\{l,r\})$. It follows that:

$$\begin{aligned}
V_1\left(A,\hat{\Sigma}_2(A,\sigma_2^*)\right) &= \min_{\sigma_2\in\Delta(\{l,r\})} U_1(A,\sigma_2) = x \\
V_1\left(L,\hat{\Sigma}_2(A,\sigma_2^*)\right) &= \min_{\sigma_2\in\Delta(\{l,r\})} U_1(L,\sigma_2) = \min_{\sigma_2\in\Delta(\{l,r\})} 3\sigma_2(l) = 0 \\
V_1\left(R,\hat{\Sigma}_2(A,\sigma_2^*)\right) &= \min_{\sigma_2\in\Delta(\{l,r\})} U_1(R,\sigma_2) = \min_{\sigma_2\in\Delta(\{l,r\})} 3\sigma_2(r) = 0
\end{aligned}$$

and

$$\begin{aligned}
V_1\left(A,\hat{\Sigma}_2(L,\sigma_2^*)\right) &= V_1\left(A,\hat{\Sigma}_2(R,\sigma_2^*)\right) = x \quad ; \quad V_1\left(L,\hat{\Sigma}_2(L,\sigma_2^*)\right) = V_1\left(L,\hat{\Sigma}_2(R,\sigma_2^*)\right) = 3\sigma_2^*(l) \\
V_1\left(R,\hat{\Sigma}_2(L,\sigma_2^*)\right) &= V_1\left(R,\hat{\Sigma}_2(R,\sigma_2^*)\right) = 3\sigma_2^*(r)
\end{aligned}$$

Since $3\sigma_2^*(l) \geq 3\sigma_2^*(r)$ if and only if $\sigma_2^*(l) \geq 1/2$, and $1/2 > x/3$, we thus have

$$\begin{aligned}
A &\in \arg\max_{s_1\in S_1} V_1\left(s_1,\hat{\Sigma}_2(A,\sigma_2^*)\right) \quad ; \quad L \in \arg\max_{s_1\in S_1} V_1\left(s_1,\hat{\Sigma}_2(L,\sigma_2^*)\right) \Longleftrightarrow \sigma_2^*(l) \geq \frac{1}{2} \\
R &\in \arg\max_{s_1\in S_1} V_1\left(s_1,\hat{\Sigma}_2(R,\sigma_2^*)\right) \Longleftrightarrow \sigma_2^*(r) \geq \frac{1}{2}
\end{aligned}$$

We conclude that

$$\sigma_1^*(L) > 0 \Longrightarrow \sigma_2^*(l) \geq \frac{1}{2} \quad ; \quad \sigma_1^*(R) > 0 \Longrightarrow \sigma_2^*(r) \geq \frac{1}{2} \tag{14}$$

On the other hand, given any $\sigma_1^* \in \Delta(\{A,L,R\})$, if players in role 2 keep playing $l$ the frequency with which they observe $A$ (they won $x$), $L$ (they won 3) and $R$ (they won 0) are, respectively, $\sigma_1^*(A)$, $\sigma_1^*(L)$ and $\sigma_1^*(R)$. Hence, $\hat{\Sigma}_1(l,\sigma_1^*) = \{\sigma_1^*\}$. Similarly, $\hat{\Sigma}_1(r,\sigma_1^*) = \{\sigma_1^*\}$. It follows that:

$$\begin{aligned}
V_2\left(l,\hat{\Sigma}_1(l,\sigma_1^*)\right) &= V_2\left(l,\hat{\Sigma}_1(r,\sigma_1^*)\right) = 3\sigma_1^*(L) + x\sigma_1^*(A) \\
V_2\left(r,\hat{\Sigma}_1(l,\sigma_1^*)\right) &= V_2\left(r,\hat{\Sigma}_1(r,\sigma_1^*)\right) = 3\sigma_1^*(R) + x\sigma_1^*(A)
\end{aligned}$$

Hence,

$$\begin{aligned}
l &\in \arg\max_{s_2\in S_2} V_2\left(s_2,\hat{\Sigma}_1(l,\sigma_1^*)\right) \Longleftrightarrow 3\sigma_1^*(L) \geq 3\sigma_1^*(R) \\
r &\in \arg\max_{s_2\in S_2} V_2\left(s_2,\hat{\Sigma}_1(r,\sigma_1^*)\right) \Longleftrightarrow 3\sigma_1^*(R) \geq 3\sigma_1^*(L)
\end{aligned}$$

We conclude that

$$\sigma_2^*(l) > 0 \Longrightarrow \sigma_1^*(L) \geq \sigma_1^*(R) \tag{15}$$
$$\sigma_2^*(r) > 0 \Longrightarrow \sigma_1^*(R) \geq \sigma_1^*(L) \tag{16}$$

Summing up:

1. Suppose $\sigma_1^*(L) > 0$ and $\sigma_1^*(R) > 0$. By (14), this implies

$$\sigma_2^*(l) \geq \frac{1}{2} \quad \text{and} \quad \sigma_2^*(r) \geq \frac{1}{2} \tag{17}$$

In turn, by (15) and (16), condition (17) holds only if $\sigma_1^*(L) = \sigma_1^*(R)$. Hence, $(\sigma_1^*, \sigma_2^*)$ with

$$\sigma_1^*(L) = \sigma_1^*(R) > 0 \quad \text{and} \quad \sigma_2^*(l) = \sigma_2^*(r) = \frac{1}{2}$$

are the WSCEs with both $\sigma_1^*(L) > 0$ and $\sigma_1^*(R) > 0$.

2. Suppose $\sigma_1^*(L) > \sigma_1^*(R) = 0$. By (16), $\sigma_1^*(L) > \sigma_1^*(R)$ implies $\sigma_2^*(r) = 0$. Hence, $(\sigma_1^*, \sigma_2^*)$ with

$$\sigma_1^*(L) > 0 \quad \text{and} \quad \sigma_2^*(r) = 0$$

are the WSCEs with $\sigma_1^*(R) = 0$.

3. Suppose $\sigma_1^*(R) > \sigma_1^*(L) = 0$. By proceeding as before, the $(\sigma_1^*, \sigma_2^*)$ with

$$\sigma_1^*(R) > 0 \quad \text{and} \quad \sigma_2^*(r) = 1$$

are the WSCEs with $\sigma_1^*(L) = 0$.

4. Suppose $\sigma_1^*(R) = \sigma_1^*(L) = 0$. By Lemma 15-(ii), $\{(A, \sigma_2^*) : \sigma_2^* \in \Delta(\{l, r\})\}$ are the WSCEs with $\sigma_1^*(A) = 1$.

We conclude that $(\sigma_1^*, \sigma_2^*)$ is a WSCE if and only if

$$\begin{aligned} \text{either } \sigma_1^*(L) &= \sigma_1^*(R) > 0 \quad \text{and} \quad \sigma_2^*(r) = \frac{1}{2} \quad \text{or} \quad \sigma_1^*(L) > \sigma_1^*(R) = 0 \quad \text{and} \quad \sigma_2^*(r) = 0 \\ \text{or } \sigma_1^*(R) &> \sigma_1^*(L) = 0 \quad \text{and} \quad \sigma_2^*(r) = 1 \quad \text{or} \quad \sigma_1^*(A) = 1 \quad \text{and} \quad \sigma_2^*(r) \in [0, 1] \end{aligned}$$

as desired. ■

Note that the WSCE are also SSCE when $\phi_1$ is concave enough (the shape of $\phi_2$ is immaterial since player 2 does not perceive any model uncertainty).

Finally, Lemma 15 and Proposition 16 allow to characterize the set of all BSCEs:

**Corollary 17** $(\sigma_1^*, \sigma_2^*)$ *is a BSCE if and only if*

$$\sigma_1^*(A) = 0 \quad \textit{and} \quad \sigma_1^*(R) = \sigma_2^*(r) \in \left\{0, \frac{1}{2}, 1\right\}$$

In this game the set of BSCEs is thus strictly included in that of SSCEs (for $\phi_1$ concave enough) and WSCEs.

## 6 Randomization

In this section we allow agents to commit to arbitrary objective randomization devices. Credible randomizations require a richer commitment technology than assumed so far. This can be seen by focussing on static games. In such games, playing a pure strategy simply means that an action is irreversibly chosen. But there is a commitment issue in playing mixed strategies. Suppose that a particular mixed strategy $\sigma_i$ is optimal. If this is true for a Bayesian agent, then also each pure strategy in the support of $\sigma_i$ is optimal, therefore $\sigma_i$ can be implemented by mapping each strategy in supp$\sigma_i$ to the realization of an appropriate roulette spin and then choosing the pure strategy associated to the realization. On the other hand, an ambiguity averse agent who finds $\sigma_i$ optimal, need not find the pure strategies in supp$\sigma_i$ optimal (within the simplex $\Delta(S_i)$). Therefore, unlike a Bayesian agent, an ambiguity averse one has to be able to irreversibly delegate his choice to the random device. Despite these reservations, we still find the extension to mixed strategies worth exploring.

When each agent in population $i$ plays a mixed strategy, a mixed strategy distribution $\varsigma_i \in \Delta(\Delta(S_i))$ obtains. To simplify the analysis, we consider only the case in which the ensuing distributions $\varsigma_i$ on mixed strategies have finite support.[36] For each role $i$, agents in population $i$ are "endogenously" partitioned into $k_i$ "epistemic types," and $\lambda_i^\ell$ is the fraction of agents in population $i$ of type $\ell = 1, ..., k_i$. In every period, agents are drawn at random and matched to play the given game. Each type $\ell$ of agent in population $i$ keeps playing some (possibly degenerate) mixed strategy $\sigma_i^\ell$. This generates a mixture distribution $\sigma_i^* = \sum_{\ell=1}^{k_i} \lambda_i^\ell \sigma_i^\ell$ on pure strategies in population $i$. Agents of type $\ell$ in population $i$ accumulate evidence (a message in every period) that in the long run is summarized by the long-run frequencies $\mu_i^{*,\ell} \in \Delta(M)$, which depend on $\sigma_i^\ell$ and $\sigma_{-i}^*$: $\mu_i^{\ell,*} = \hat{F}_i(\sigma_i^\ell \times \sigma_{-i}^*)$. As before, agents are informally assumed to be long-run empiricists: for every observable event, they equate its probability with its observed long-run frequency, hence the set of possible models for agents of type $\ell$ is $\hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)$. In the following definitions, we give stability conditions for a profile of strategy distributions $(\sigma_i^*)_{i\in I}$ interpreted as mixtures. We use superscript $\rho$ to denote the equilibrium concepts obtained when randomization is allowed.

**Definition 18** *A profile of mixed strategies $(\sigma_i^*)_{i\in I}$ is a* smooth selfconfirming equilibrium with randomization (SSCE$^\rho$) *of $(\Gamma, f, \phi)$ if, for each $i$, there are a positive integer $k_i \in \mathbb{N}$, a vector of weights $(\lambda_i^1, ...\lambda_i^{k_i}) \in \mathbb{R}_{++}^{k_i}$ with $\sum_{\ell=1}^{k_i} \lambda_i^\ell = 1$ and a vector of mixed strategies $(\sigma_i^1, ..., \sigma_i^{k_i}) \in [\Delta(S_i)]^{k_i}$, such that*

1. *(aggregation) $\sigma_i^* = \sum_{\ell=1}^{k_i} \lambda_i^\ell \sigma_i^\ell$,*
2. *(*confirmed rationality*) for each $\ell = 1, ..., k_i$ there is a belief $p_i^\ell \in \hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)$ with*

$$\sigma_i^\ell \in \arg\max_{\sigma_i \in \Delta(S_i)} \int_{\hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)} \phi_i(U_i(\sigma_i, \sigma_{-i}))dp_i^\ell(\sigma_{-i})$$

*A profile $(\sigma_i^*)_{i\in I}$ satisfying condition 1 above is a* Waldean selfconfirming equilibrium (WSCE$^\rho$) *of $(\Gamma, f)$ if, for each $i$ and $\ell$*

$$\sigma_i^\ell \in \arg\max_{\sigma_i \in \Delta(S_i)} \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)} U_i(\sigma_i, \sigma_{-i}),$$

[36] We conjecture that, for generic payoffs, this is enough to characterize equilibrium distributions of outcomes.

*and it is a* Bayesian selfconfirming equilibrium with randomization (SSCE$^\rho$) *of* $(\Gamma, f)$ *is a SSCE$^\rho$ of the game with* $(\Gamma, f, \phi)$ *with ambiguity neutral players, where each* $\phi_i$ *is the identity. A smooth (Waldean, Bayesian) equilibrium profile is* symmetric *if* $k_i = 1$ *for each* $i \in I$*.*

Since for Bayesian agents the value of an optimal mixed strategy is the same as the value of any pure strategy in its support, the BSCE concept give rise to the same strategy distributions as the BSCE$^\rho$ concept:

$$BSCE = BSCE^\rho.$$

Indeed, BSCE is a special case of BSCE$^\rho$ where agents do not randomize, and a BSCE$^\rho$ is equivalent to a BSCE where agents of type $\ell$ in population $i$ are further divided into "subtypes" playing the pure strategies in the support of $\sigma_i^\ell$.

A symmetric BSCE$^\rho$ is such that every pure strategy $s_i^*$ played with positive probability in equilibrium can be justified by the same confirmed belief $p_i$. Thus, the symmetric BSCE$^\rho$ concept is equivalent to the SCE with unitary beliefs of Fudenberg and Levine (1993a) generalized to arbitrary feedback.

The comparative statics results and relationships of Section 4 extend seamlessly to equilibria with randomization (the same arguments apply almost *verbatim* with pure strategies replaced by mixed strategies):

- the coarser the feedback, the larger the set of equilibria of each kind,
- under observable payoffs, the set of SSCE$^\rho$s becomes larger as ambiguity aversion increases,
- under observable payoffs and ambiguity aversion,

  $$BSCE^\rho \subseteq SSCE^\rho \subseteq WSCE^\rho;$$

  furthermore, the first inclusion holds in every two-person game with any feedback and ambiguity attitudes.

Allowing for randomization reduces the scope for differences between the BSCE, SSCE and WSCE concepts. Indeed, in two-person games with observable payoffs and ambiguity aversion they coincide.

**Proposition 19** *In two-person games, every WSCE$^\rho$ is also a SSCE$^\rho$ and BSCE$^\rho$. Hence, in two-person games with observable payoffs and ambiguity aversion the three kinds of self-confirming equilibrium with randomization coincide:*

$$BSCE^\rho = SSCE^\rho = WSCE^\rho.$$

**Proof** Fix a WSCE$^\rho$ $\sigma^*$. For each $i \in I$ and $\ell \in \{1, ..., k_i\}$, $\hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)$ is nonempty and compact, since $-i$ is a single player it is also convex. Choose

$$\bar{\sigma}_{-i}^\ell \in \arg \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)} U_i(\sigma_i^\ell, \sigma_{-i}).$$

By definition of WSCE$^\rho$

$$\sigma_i^\ell \in \arg\max_{\sigma_i \in \Delta(S_i)} \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)} U_i(\sigma_i, \sigma_{-i})..$$

Hence, the maxmin theorem implies

$$\sigma_i^\ell \in \arg\max_{\sigma_i \in \Delta(S_i)} U_i(\sigma_i, \bar{\sigma}_{-i}^\ell).$$

Thus, $\bar{\sigma}_{-i}^\ell$ is a confirmed degenerate belief justifying $\sigma_i^\ell$. This shows that $\sigma^*$ is also a BSCE$^\rho$ and a SSCE$^\rho$ (ambiguity attitudes do not matter if $p_i^\ell$ is a Dirac measure). Assuming observable payoffs and ambiguity aversion, we have both $BSCE^\rho \subseteq SSCE^\rho \subseteq WSCE^\rho$ and $WSCE^\rho \subseteq BSCE^\rho$, therefore $BSCE^\rho = SSCE^\rho = WSCE^\rho$. ■

The game depicted in Figure 3 can be used to illustrate how WSCE differs from WSCE$^\rho$ (and BSCE), because we may have

$$\max_{s_i \in S_i} \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U_i(s_i, \sigma_{-i}) < \max_{\sigma_i \in \Delta(S_i)} \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U_i(\sigma_i, \sigma_{-i}).$$

Similar considerations apply to SSCE and SSCE$^\rho$.

**Example 20** *Consider again the common interest game in Figure 3 of the Introduction, assuming perfect feedback. Strategy A is strictly dominated by mixed strategy* $\frac{1}{2}L + \frac{1}{2}R$*, which yields expected utility* $\frac{3}{2}$ *irrespective of 2's strategy. Therefore, outcome A cannot be induced by any selfconfirming equilibrium with randomization (see Proposition 19). But, as already argued, A is a WSCE outcome: if player 1 chooses A he cannot observe anything about player 2 and his set of confirmed conjectures is* $\hat{\Sigma}_1(A, \sigma_2) = \Delta(\{\ell, r\})$ *for every* $\sigma_2$*; the minimum EU value of L and R is* 0*, the value of A is* 1*. In particular, one can show that if* $\phi_1$ *is concave*

$$WSCE^\rho = SSCE^\rho = BSCE^\rho = NE = \left\{\sigma^* : \sigma_1^*(A) = 0, \sigma_1^*(R) = \sigma_2^*(r) \in \left\{0, \frac{1}{2}, 1\right\}\right\}$$

*As observed in Section 5,* $WSCE = SSCE$ *if* $\phi_1$ *is sufficiently concave.* ▲

Next we turn to the relationship between selfconfirming equilibria with randomization and Nash equilibrium. We know that $NE \subseteq BSCE = BSCE^\rho$ in all games, $BSCE^\rho \subseteq SSCE^\rho$ in two-person games, $BSCE^\rho \subseteq SSCE^\rho \subseteq WSCE^\rho$ in games with observable payoffs and ambiguity aversion, and analogous relations hold for symmetric equilibria. The following proposition provides a partial converse. Consistently with our notation, we let $\hat{\zeta}(\sigma)$ denote the random path induced by the product measure $\times_{i \in I} \sigma_i$.

**Proposition 21** *In games with two players and perfect feedback, every symmetric WSCE*$^\rho$ *is realization equivalent to some mixed Nash equilibrium; therefore*

$$\hat{\zeta}(symWSCE^\rho) = \hat{\zeta}(symBSCE^\rho) \subseteq \hat{\zeta}(NE).$$

*Furthermore, under ambiguity aversion*

$$\hat{\zeta}(symWSCE^\rho) = \hat{\zeta}(symSSCE^\rho) = \hat{\zeta}(symBSCE^\rho) \subseteq \hat{\zeta}(NE).$$

**Proof** Battigalli (1987) proved that, in two-person games with perfect feedback, a symmetric BSCE$^{\rho}$ (a selfconfirming equilibrium with unitary beliefs) is realization-equivalent to a mixed Nash equilibrium. Then, the result follows from Proposition 19. ■

**Example 22** *Consider the "horse" game with three players and imperfect information depicted in Figure 2. Path $(A, a)$ (for any feedback structure) is supported by (Waldean, Bayesian, smooth) symmetric selfconfirming equilibria with randomization, but it is not a pure or mixed Nash equilibrium path. This shows that the assumptions of the two propositions above are quite tight.*[37] ▲

# 7 Discussion

Let us take stock of what we did so far. We analyzed a notion of SCE with agents who have non-Bayesian attitudes toward uncertainty on the true data generating process they are facing. We already argued that, as in the traditional notion of SCE, this uncertainty comes from a partial identification problem: the mapping from strategy distributions to the distributions of observations available to an agent is not one to one. This is discussed and illustrated in the context of decision problems (Section 2) before we move to the game theoretic analysis. We used as our workhorse the KMM smooth-ambiguity model, which separates beliefs from ambiguity attitudes and is therefore particularly well suited to connect with the previous literature on SCE. Our main equilibrium concept, smooth selfconfirming equilibrium (SSCE) admits traditional Bayesian SCE as a special case and Waldean (maxmin) SCE as a limit case. According to our main result, the set of SSCE expands when agents become more ambiguity averse (assuming observability of payoffs). This allows us to derive intuitive relationships between different versions of SCE and show that Nash equilibrium is a refinement of all of them, which guarantees existence. Equivalence results are provided under specific assumptions on the information structure. In the core sections of the paper (Sections 3.2 and 4) we assume that agents play the strategic form of a dynamic game and can only choose pure strategies. Mixed strategies are considered in a separate section where we put forward notions of equilibrium with randomization: on the one hand, our main results extend seamlessly to carry over to these equilibrium concepts; on the other hand, we obtain equivalence results under weaker assumptions.

We now discuss the following points: (1) We avoided dynamic consistency issues assuming that agents play the strategic form of the game. How should we defined our equilibrium concepts without this commitment assumption? How are our results going to be affected? (2) We clarified that our notions of SCE make sense under minimal assumptions about agents' knowledge of the game. But under common knowledge of the game, stronger notions of equilibrium should arguably be considered. (3) Like Fudenberg and Levine (1993a,b), we assume that each agent has only access to his "personal database" of experiences in the games he played. How is the SSCE concept affected if agents have access to a "public

[37] The same holds for symmetric Bayesian selfconfirming equilibria supported by beliefs that satisfy independence across opponents. This can happen because the "horse" does not feature observable deviators. On the other hand, in all games with perfect feedback and observable deviators, every such equilibrium is realization equivalent to some Nash equilibrium. This can be shown directly. Kamada (2010) proves it as a corollary of another result.

database"? (4) Fudenberg and Levine (1993b) give a "steady state learning" foundation to the SCE concept, is it possible to do the same for SSCE?

## 7.1 Dynamic consistency

When agents really play an extensive-form game, not its strategic form, they cannot commit to any strategy. A strategy for an agent is just a plan that allows him to evaluate the likely consequences of taking actions at any information set. The plan is credible and can be implemented only if it prescribes actions that have the highest value, given the agent's beliefs and planned continuation. Plans with this *unimprovability* property can be obtained by means of a backward induction procedure on the subjective decision tree implied by the agent's beliefs. Such a procedure may have to break ties. How ties are broken is immaterial for a Bayesian agent, but matters for a non Bayesian one. Consider two information sets of player $i$, viz. $\bar{h}_i$ and $h_i$, so that $\bar{h}_i$ immediately precedes $h_i$ in the tree of information sets of $i$, and let $\alpha_i(\bar{h}_i, h_i)$ be the action at $\bar{h}_i$ that leads to $h_i$.[38] If ties at $h_i$ are broken so as to maximize the value of action $\alpha_i(\bar{h}_i, h_i)$ (and further ties are broken to maximize the value of $\alpha_i(\hat{h}_i, \bar{h}_i)$, where $\hat{h}_i$ is the immediate predecessor of $\bar{h}_i$, and so on), the resulting strategy satisfies *consistent planning*. We can make this precise in the context of the smooth-ambiguity model, and thus provide notions of SSCE assuming unimprovability, or consistent planning. For brevity, we write SSCE$^\delta$ to refer to such smooth selfconfirming equilibria with dynamic consistency. What difference does dynamic consistency make?

It is well known that strategies that an ambiguity averse agent would commit to, if he could, need not be unimprovable.[39] Therefore, SSCE$^\delta$ is not equivalent to SSCE. Furthermore, it is not obvious whether our main comparative statics result is valid with this definition. We can offer examples and intuitions based on our preliminary analysis of this issue.

**Example 23** *The game* $\Gamma_w$ *given by*

$$\begin{array}{ccc} 1 & \xrightarrow{\text{in}} & 0 & \xrightarrow{\text{c } (\frac{1}{2})} \\ \downarrow \text{out} & & \downarrow \text{s } (\frac{1}{2}) & \\ 1.7 & & 2 & \end{array}$$

| 1\{2,3} | L,l | R,l | L,r | R,r |
|---|---|---|---|---|
| T | 4 | 0 | 0 | 0 |
| M | 0 | 0 | 0 | 4 |
| B | $w$ | $w$ | $w$ | $w$ |

Figure 4

*is a three-person, common interest game with a chance move.*[40] *The objective* $\frac{1}{2} : \frac{1}{2}$ *probabilities of this move are known. Parameter* $w > 1$ *is the "Waldean value" of the subgame with root* (in, c)*. Assume that the agent in role* 1 *has the following correlated belief:*

$$p_1 = \left(\frac{1}{2}\delta_{\mathrm{c}} + \frac{1}{2}\delta_{\mathrm{s}}\right) \times \left(\frac{1}{2}\delta_{\mathrm{L}} \times \delta_{\mathrm{l}} + \frac{1}{2}\delta_{\mathrm{R}} \times \delta_{\mathrm{r}}\right),$$

[38] Perfect recall implies that function $\alpha_i(\cdot, \cdot)$ is well defined, and that the collection of information sets of $i$ (plus the singleton containing the root of the game, and with the obvious precedence relation derived from precedence between nodes) is a tree.

[39] Siniscalchi (2011) reports examples and provides an in-depth analysis of dynamic consistency under ambiguity aversion.

[40] Numbers at terminal nodes, including the boxes in the matrix subgame, give the common payoff, chance is labeled 0.

*Intuitively, he thinks that all the agents playing in roles* 2 *and* 3 *"attended the same school" and hence are doing the same thing, but he does not know what. Suppose his ambiguity attitudes are represented by* $\phi_1(\cdot) = \sqrt{\cdot}$. *On the one hand, actions* T *and* M, *once the matrix game is reached, have value* $\left(\frac{1}{2}\sqrt{4}+\frac{1}{2}\sqrt{0}\right)^2 = 1$. *On the other hand, from the perspective of the agent at the root of* $\Gamma_w$, *the value of committing to strategy* in.T *(or* in.M*) is* $\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{1}\right)^2 = 1+\frac{\sqrt{3}}{2} > 1.7$,[41] *and strategy* in.B *has value* $\frac{1}{2}2+\frac{1}{2}w = 1+\frac{w}{2}$. *If* $1 < w < 1.4$, *the unique dynamically consistent strategy is* out.B, *which is also an* $SSCE^{\delta}$ *supported by belief* $p_1$, *for any* $f$, $\phi_2$, *and* $\phi_3$. *But, under perfect feedback, there is no SSCE (with commitment) supported by such beliefs: the set of best response to* $p_1$ *is* $\{\text{in.T}, \text{in.M}\}$, *but playing* in *reveals the true strategy distributions of agents playing in roles* 2 *and* 3 *(*out *is supported as an SSCE with commitment under different beliefs).* ▲

The following example shows that a naive extension of our main result, Theorem 9, to $\text{SSCE}^{\delta}$ does not hold. Specifically, there are games $(\Gamma, f, \phi)$ and $(\Gamma, f, \bar{\phi})$, with the former more ambiguity averse than the latter, and a profile of strategy distributions $\sigma^*$ that is a $\text{SSCE}^{\delta}$ of $(\Gamma, f, \bar{\phi})$ but not a $\text{SSCE}^{\delta}$ of $(\Gamma, f, \phi)$.

**Example 24** *The game* $\Gamma$ *given by*

$$\begin{array}{c} 1 \\ \downarrow \text{out} \\ 2 \end{array} \quad \xrightarrow{\text{in}}$$

| $1\backslash\{2,3\}$ | L, l | R, l | L, r | R, r |
|---|---|---|---|---|
| T | 4 | 0 | 0 | 0 |
| M | 0 | 0 | 0 | 4 |
| B | 2 | 2 | 2 | 2 |

Figure 5

*is a three-person, common interest game. It can be checked that* out.T *is an* $SSCE^{\delta}$ *strategy if player 1 is Bayesian. There is exactly one belief supporting* out.T *as an unimprovable strategy for a Bayesian agent:*

$$p_1^* = \frac{1}{2}(\delta_{\text{L}} \times \delta_{\text{l}}) + \frac{1}{2}(\delta_{\text{R}} \times \delta_{\text{r}}).$$

*To see, first note that* out.T *is unimprovable under any belief* $p_1$ *if and only if* $\bar{p}_1(\text{L},\text{l}) = \frac{1}{2} = \bar{p}_1(\text{R},\text{r})$*; indeed, if* $\max\{\bar{p}_1(\text{L},\text{l}), \bar{p}_1(\text{R},\text{r})\} > \frac{1}{2}$, *then the best response at the root is* in, *if* $\max\{\bar{p}_1(\text{L},\text{l}), \bar{p}_1(\text{R},\text{r})\} \leq \frac{1}{2}$ *and* $\min\{\bar{p}_1(\text{L},\text{l}), \bar{p}_1(\text{R},\text{r})\} < \frac{1}{2}$, *then the unique best reply in the subgame is* B. *Next observe that player 1 is certain that the objective distribution of strategy pairs of* $\{2,3\}$ *is a product measure, i.e.,* $p_1 \in \Delta(\Delta(S_2) \bigotimes \Delta(S_3))$. *Now, consider* $\phi$ *and* $\bar{\phi}$ *with* $\bar{\phi}_1 = \text{Id}_{\mathbb{R}}$, $\phi_1$ *strictly concave and* $\phi_j = \bar{\phi}_j$ *for* $j = 2,3$. *Strategy* out.T *is not unimprovable for* $(p_1^*, \phi_1)$ *because the value of* T *is* $\phi_1^{-1}\left(\frac{1}{2}\phi_1(4)+\frac{1}{2}\phi_1(0)\right) < 2$ *(by the strict concavity of* $\phi_1$*). More generally, for every* $p_1$ *such that* $\bar{p}_1(L,l) \leq \frac{1}{2}$, *the* $\phi_1$*-value of* T *is strictly less than 2 (let* $\Lambda = \sigma_2(L)$, $\lambda = \sigma_3(l)$, *then the value of* T *is*

$$\phi_1^{-1}\left(\int_{[0,1]^2} \phi_1(4\Lambda\lambda) p_1(\text{d}\Lambda \times \text{d}\lambda)\right) < \int_{[0,1]^2} 4\Lambda\lambda p_1(\text{d}\Lambda \times \text{d}\lambda) = 4\bar{p}_1(\text{L},\text{l})$$

[41] Here is where the numbers under $\sqrt{\cdot}$ come from: $3 = U_1(\text{in.T}, \sigma_0 \times \delta_{\text{L}} \times \delta_{\text{l}})$, $1 = U_1(\text{in.T}, \sigma_0 \times \delta_{\text{r}} \times \delta_{\text{r}})$, with $\sigma_0 = \frac{1}{2}\delta_{\text{c}} + \frac{1}{2}\delta_{\text{s}}$.

*by the strict concavity of* $\phi_1$*, whenever* $p_1$ *is not a Dirac measure; if* $\bar{p}_1(\mathrm{L},\mathrm{l}) \leq \frac{1}{2}$ *then the value of* T *is less than* 2*). Therefore,* out.T *is a SSCE*$^\delta$ *strategy of* $(\Gamma, f, \bar{\phi})$*, with* $\bar{\phi} = (Id_{\mathbb{R}}, \phi_2, \phi_3)$*, but not of the more ambiguity averse game* $(\Gamma, f, \phi)$*. Letting* $\phi_2 = \phi_3 = Id_{\mathbb{R}}$*, this also shows that a BSCE*$^\delta$ *need not be a SSCE*$^\delta$*.*

*Although* out.T *is not a SSCE*$^\delta$ *strategy in the more ambiguity averse game* $(\Gamma, f, \phi)$*, the realization-equivalent strategy* out.B *is. We conjecture that the following version of the comparative statics result holds: if* $\sigma^*$ *is a SSCE*$^\delta$ *of* $(\Gamma, f, \bar{\phi})$ *and* $(\Gamma, f, \phi)$ *is more ambiguity averse than* $(\Gamma, f, \phi)$*, then there is a SSCE*$^\delta$ *of* $(\Gamma, f, \phi)$ *realization-equivalent to* $\sigma^*$*.* ▲

There is another problem related to dynamic consistency: as in Fudenberg and Levine (1993b), the notion of best reply used in this paper allows for suboptimal behavior at unexpected information sets. For example, in the game of Figure 1, *Right* is strictly dominated by *Left* conditional on *Down*, but there is a set selfconfirming equilibria (for arbitrary $\phi$) where all agents in role 1 play *Across* and the equilibrium distribution $\sigma_2$ is arbitrary. To deal with this problem we can represent players' beliefs as *conditional probability systems*: an agent in role $i$ have an initial belief $p_i$, for each information set $h_i$ he would hold a corresponding conditional belief $p_i(\cdot|h_i)$ over the strategy distributions of the opponents, and these beliefs are related to each other *via* Bayes rule whenever possible (see, for example, Battigalli and Siniscalchi, 1999). With this, we can give stronger versions of unimprovability and consistent planning to obtain a *refined* notion of SSCE$^\delta$. In the game of Figure 1, the only refined equilibrium strategy for agents in role 2 is *Left*. However, *Across* is still an equilibrium strategy because an agent playing *Across* may incorrectly believe that *Right* has positive probability; thus, the set of equilibrium outcomes is the same.[42]

An agent in role 1 may believe that the irrational action *Right* has positive probability for the simple reason that he need not know the utility function of 2. Under complete information (that is, common knowledge of the game) further refinements are worth exploring. Clearly, the unique selfconfirming equilibrium of the gama of Figure 1 consistent with belief on the co-player's rationality under complete information is the symmetric equilibrium $(D, L)$.

## 7.2 Rationalizable selfconfirming equilibrium

In a selfconfirming equilibrium agents are rational and their beliefs are confirmed. Assuming that the game[43] is common knowledge, it is interesting to explore the implications of assuming, on top of this, that there is common certainty (probability-one belief) of rationality and confirmation of beliefs. Interestingly, the set of *rationalizable SCE* thus obtained may be a strict subset of the set of SCEs consistent with common certainty of rationality, which in turn may be a strict subset of the set of SCEs as shown in the example above (see Rubinstein and Wolinsky (1994), Dekel et al (1999), Battigalli (1999), Battigalli and Siniscalchi (2002, 2003), Esponda (2011), Fudenberg and Kamada (2011), and Letina (2011). The separation between ambiguity attitudes and beliefs in the smooth-ambiguity model allows a relatively straightforward extension of this idea to obtain a notion of rationalizable SSCE.

[42] The result is more general: for every SSCE$^\delta$ there is a realization equivalent refined-SSCE$^\delta$. This requires a natural "ex post perfect recall" assumption that we did not have to make explicit so far: for each $i$ and $m$, all $z \in f_i^{-1}(m)$ determine the same sequence of actions and information sets of $i$.

[43] Or, at least, a part of the game such as $(\Gamma, f)$.

### 7.3 Selfconfirming equilibrium with a public database

The SSCE concept defined in this paper, like Fudenberg and Levine's SCE, rests on the assumption that agents have only access to their personal experiences. How does this affect our results? The answer is quite straightforward if, like Fudenberg and Levine, we assume perfect feedback. In this case, the "public database" induced a by profile of strategy distributions $\sigma$ is $\hat{\zeta}(\sigma) \in \Delta(Z)$. The key observation is that the "personal database" of an agent playing $s_i \in \mathrm{Supp}\sigma_i$ is subsumed by the "public database": personal experiences allow to learn the conditional frequencies of opponents' actions at the opponents' information sets visited with positive frequency under $(s_i, \sigma_{-i})$, a sub-collection of the collection of opponents' information sets visited with positive frequency under $(\sigma_i, \sigma_{-i})$. Therefore $\sigma^*$ is a *public SSCE* if for each $i$ and $s_i^* \in \mathrm{Supp}\sigma_i^*$ there is some $p_i \in \hat{\Sigma}_{-i}(\sigma^*) = \{\sigma_{-i} : \exists \sigma_i \in \Delta(S_i), \hat{\zeta}(\sigma_i, \sigma_{-i}) = \hat{\zeta}(\sigma^*)\}$ such that $s_i^*$ is a smooth best response to $p_i$. (It can be shown that this is equivalent to a definition of *SSCE with unitary beliefs* whereby the justifying belief $p_i$ must be the same for all strategies in $\mathrm{Supp}\sigma_i^*$.) Our results can be adapted to this definition of public SSCE. If perfect feedback does not hold, the analysis is more complex. First, it is not obvious what a "public database" is supposed to be. Second, for some definitions of "public database", the personal database need not be subsumed by the public one. Different definitions of SSCE look plausible and it is not yet clear to us which of our main results can be adapted to such definitions.

### 7.4 Learning and steady states

Fudenberg and Levine (1993,b) provide a learning foundation of selfconfirming equilibrium within an overlapping generation model with stationary aggregate distributions. The stationarity assumption is a clever trick that allows to use consistency and convergence results in Bayesian statistics about sampling from a "fixed urn" of unknown distributions. It would be important to extend such results to the smooth-ambiguity model and SSCE, and to use results on updating of sets of priors to obtain an analogous learning foundation for the WSCE concept proposed here. Fudenberg and Levine (2006) obtain a refinement of SCE based on the idea of steady state learning with extremely patient and long-lived agents and illustrate it with an application. Similar ideas are worth exploring in the current framework.

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