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Can Monetary Policies Inflate a Stock Market Bubble? A Regime Switching Model of Periodically Collapsing Bubbles

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Abstract

We study whether and how monetary policymakers may have contributed to inflate asset price bubbles and in general what are the potentially complex, non-linear linkages between short-term policy rates and the size and expected durations of equity bubbles. In particular, we extend empirical models of periodically collapsing, rational bubbles to test whether and to what extent the long cycle of rates at the zero lower bound and of quantitative easing policies may have increased the probability of bubbles inflating and persisting, with special emphasis on the US stock market. We find that the linkages between S&P returns and rate-based indicators of monetary policies contain evidence of recurring regimes that can be characterised as one of a persisting vs. one of a collapsing bubble. Moreover, the probabilities of financial markets transitioning from a bubble to a state of (partial) collapse turns out to depend on both the initial, relative size of the bubble and on monetary policy indicators. This implies that an easier (tighter) monetary policy will inflate (deflate) a bubble through a simple, regression-style effect, but also yield a non-linear, "concave" effect by which sufficiently low (high) rates are enough for a bubble to inflate (deflate) with high probability. Besides fitting the data, the resulting, parsimonious, regime switching models provide an accurate and economically valuable predictive performance, even when transaction costs are taken into account.

Keywords: Rational bubbles, monetary policy, stock returns, regime switching, forecasting.

JEL codes: G12, E52, C58, G17.

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1 Introduction

Since the seminal papers by Blanchard and Watson (1982) and Diba and Grossman (1988), it is well known that asset prices may show a tendency to rise substantially over protracted periods posing a systematic and increasing deviation from their "fundamental value" (defined as the risk-adjusted present value of all expected future cash flows). This increase is usually followed by a very quick drop in manners which are completely rational and therefore consistent with market efficiency: in such episodes, called of *rational* speculative bubbles, investors may act rationally by continuing to pay ever-further-inflated prices, since they were being compensated for it.[1] This means that any empirically observed deviation of market prices from fundamentals does not have to be necessarily imputed to behavioural biases or mispricing caused by psychological forces, as these may be consistent with rational behaviour and rationally formed expectations.

In fact, a literature exists (see, e.g., Roubini, 2006, and references therein) that has shown that bubbles that are growing excessively large often lead to economic and investment distortions that may be dangerous and likely to eventually trigger bubble bursts whose real and financial consequences are severe in real terms. As the experience of the Great Financial Crisis of 2007-2008 has taught us, they may even lead to outright episodes of financial instability and panics. One view exists that optimal monetary policy should pre-emptively deal with asset bubbles—*in primis*, by avoiding that they inflate or at least by limiting their relative size using classical monetary policy instruments—rather than just mop up the damages that they cause after they burst. Exactly for this reason, even though the mechanics by which rational bubbles arise remains an active topic of research (see the discussion in Vogel, 2018), it is of great importance to formulate and test *models for how bubbles inflate and collapse*, also with the goal of isolating which drivers and market conditions may be more favourable to bubbles growing and hence to their welfare-damaging collapse becoming likely.

In recent years, many market analysts, industry experts and, at least occasionally, academic researchers, have taken turns in blaming the occurrence and persistence of alleged stock valuation bubbles on the Federal Reserve's policy decisions. Among many others, in 2010 the

[1] Rational bubbles arise because of the indeterminate aspect of the solutions to rational expectations models in asset pricing, which in the case of stock prices is implicitly reflected in the Euler equation. The price that an investor is ready to pay depends on the price that she is expected to obtain at some point in the future but such an expectation depends on the price expected even further in the future. Therefore, the Euler equation simply determines a sequence of prices but fails to "pin down" a unique price level unless somewhat arbitrarily one (the modeller) imposes some terminal condition (the transversality condition) to obtain the unique solution. However, in general the Euler equation does not rule out the possibility that the price may contain an explosive bubble, which is, as a result, fully rational. See Campbell et al. (1997) for a textbook-level introduction.

then Federal Reserve Chairman, Ben Bernanke, recognised that a few commentators had claimed that:

> *"(…) excessively easy monetary policy by the Federal Reserve in the first half of the decade helped cause a bubble in house prices in the United States, a bubble whose inevitable collapse proved a major source of the financial and economic stresses of the past two years. Proponents of this view typically argue for a substantially greater role for monetary policy in preventing and controlling bubbles in the prices of housing and other assets."* (Bernanke, 2010).[2]

Among such pundits there were journalists as well as famous colleagues of Chairman Bernanke, e.g., Roubini (2006) and Taylor (2013), who claimed that by keeping key policy rates "too low and for too long", the Federal Reserve would have made the financial crisis possible, if not even more likely than it would have otherwise been.[3]

In this paper, we take this debate seriously and we perform a variety of empirical tests of the hypothesis that monetary policy may affect the extent and probability of collapse (or equivalently, their expected duration) of rational bubbles in the US stock market. Our tests are informed by the maximum likelihood estimates of regime switching regression models characterised by time-varying transition probabilities. These are driven by a logistic function that is posited to capture the effect of a number of factors (such as the relative size of any previously existing bubble, a monthly indicator of percentage abnormal trading volume and a measure of sentiment popularised by Baker and Wurgler, 2004, orthogonalized to a range of key macroeconomic indicators). The structure of the model receives partial micro-foundation from the application of a Taylor expansion to a baseline asset pricing equation when the potential existence of a rational bubble and a non-zero probability of its collapse are considered (see Appendices A and B for proof).

Our empirical framework refines and extends earlier work on the estimation of regime switching models by Anderson, Brooks and Katsaris (2010), Brooks and Katsaris (2005a, b) and Schaller and van Norden (2002) (also in the context of REITs and housing markets, see e.g., Anderson et

[2] This quote and considerable, related commentary by high officials at the Federal Reserve during the early 2010s were in fact ending the era of what one could call the "Greenspan's standard". Such a system of beliefs was on the axiom that "(…) given our current state of knowledge, I find it difficult to envision central banks successfully targeting asset prices any time soon. However, I certainly do not rule out that future work could improve our understanding of asset price behavior, and with it, the conduct of monetary policy." (see Greenspan, 2005).

[3] These suspicions have re-surfaced recently. In the financial press, see, e.g., "Fed Risks Stoking Financial Bubble in Drive to Lift Inflation", by R. Miller, *Bloomberg*, April 3, 2019; "The Federal Reserve is the cause of the bubble in everything", by M. Howell, *The Financial Times*, January 16, 2020; "Is the bull market about to turn into a bubble?" *The Economist*, March 11, 2024.

al., 2011 and Nneji et al., 2013 and of commodities, see e.g., Brooks et al., 2015) in a precise way: we introduce the short-term rate—in its turn, captured by either the Federal Funds rate (henceforth, FFR), or Wu and Xia's (2016) US shadow rate—in both the regime-specific conditional mean functions (regressions) and in the logistic function driving the time-varying probabilities of a bubble collapsing. In this model, bubbles are stochastic and may either survive or collapse. This implies that stock market returns come from two distinct regimes, one of which corresponds to surviving bubbles and the other to collapsing bubbles. In addition, because the probability of collapse depends on a set of factors, switches in regime will be forecastable using such predictors. Besides the empirical realism of such a model extension, we provide a role to the short-term rate as this is heavily influenced by monetary policy decisions and therefore its presence in the models allows us to investigate the effects of central bank actions on the size of bubbles and on their duration. In particular, we formulate an array of hypotheses concerning the specific impact of monetary policy on bubbles and on the very persistence of bubbles (i.e., whether these are locally stable but globally unstable or both locally and globally stable) that are set to depend on the estimated sign and significance of specific coefficients that appear in a parametric, non-linear model.

Using monthly data for a long, 1954-2023 sample, our main empirical finding is that monetary policy has historically affected the relative size and duration of equity bubbles in the US. On the one hand, a higher short-term policy rate reduces expected stock returns and hence the size of any ongoing bubble irrespective of the regime in which the markets may start from on a given month. On the other hand, the effect of a higher short-term rate on the transition probabilities of the system is complex and non-linear because the data show a strong appetite for higher-order terms (powers) in the specification of the logistic transition probability function. Starting from a zero-policy rate, a higher rate does reduce the chances of a bubble collapsing, thus increasing its duration; in this respect, as claimed by Taylor (2013), very (excessively) low rates do indeed inflate bubbles. Yet, as the short-terms rates grow further and higher, the quadratic term in the logistic function is estimated to be negative and large and this leads to bubbles to burst, which means that sufficiently high rates cause bubbles to collapse. Our maximum-likelihood estimates indicate that starting from zero, Federal Fund rates raised up to approximately 2 percent, increase the duration of a bubble, similarly to the effect discussed in Galí and Gambetti (2014); past 2 percent, higher FFRs reduce its duration and in fact rates of 4 percent or higher cause almost deterministically a bubble to burst.

A similar, nonlinear dynamics also characterises the impact of the initial, relative size of bubbles on their duration: for "normal" values of the FFR in the range 1-4 percent, for relative bubble sizes exceeding approximately 80% (in absolute terms) of total equity market valuation, the expected duration of a bubble is essentially zero, indicating that large bubbles seed their own demise; yet for lower relative bubble sizes, especially in the 20-60% range, such expected duration may be high, in the order of several years. A small non-linear effect depresses the probability of bubbles surviving when they are initially small (below 20% in relative terms), but their expected duration remains in the order of several months. Overall, when the FFR is not set to exceed 4 percent (either exogenously or because monetary policy were to react to the bubble, a case on which we remain agnostic in policy terms), as a bubble gets relatively large, it is likely to persist considerably and get even larger, especially when the FFR is relatively low (between 1 and 2 percent in annualised terms). In spite of these rich non-linear effects, the estimated regime switching models deliver a time-varying system that is globally stable and features a sequence of bubbles inflating and collapsing over time.

Interestingly, such a dynamics occurs within a completely rational framework and in the absence of first-order arbitrage opportunities, in which the very mispricing caused by the bubbles represent coherent deviations of equilibrium prices from fundamentally-driven ones. These results are robust to a range of robustness checks. In particular, when we replace the FFR with the shadow rate, our main empirical insights remain intact. Likewise, when the measurement of bubbles is replaced by the methodology proposed by Campbell and Shiller (1987) that account for the fact that any deviations between observed prices and (present value) dividend-discounted ones may be predictable, our main results go through intact. Moreover, we experiment with alternative definitions of market sentiment and of abnormal relative trading volume finding essentially unchanged results.

Following the lead of Brooks and Katsaris (2005a), we also test the comparative out-of-sample predictive power of regime switching models of periodically collapsing bubbles, finding very encouraging results. When the best fitting models that have emerged in Sections 4 and 5 are applied in a rather classical, recursive OOS exercise over a 2000-2023 sample, we find that—irrespective of the loss function assumed and whether this had a statistical or economic nature—the regime switching models always outperform a number of benchmarks, such as the recursive sample mean of S&P 500 returns and single-state regressions that feature bubble indicators. Yet, the specific loss function adopted turns out to be of more relevance to discriminate between models that account for the impact of monetary policy on regime switching and those that do not. The former type of model performs the best when we assume an absolute value loss function or Sharpe ratio maximisation under non-negligible aversion to risk within a mean-variance framework.

The rest of this paper is organised as follows. Section 2 presents our research design and provides details on the structure of the empirical model to be estimated in Sections 4 and 5. Besides the estimation strategy, this section also lays down the key parametric hypotheses to be tested later on. Section 3 describes the data. Section 4 reviews the key findings concerning the simple benchmarks without regimes or with constant transition probabilities and then moves on to describe our key results for regime switching models with time-varying transition probabilities. Section 5 performs robustness checks and shows that our results are essentially unaffected by even major changes in the framework of analysis. Section 6 investigates the out-of-sample forecasting performance of the models developed under section 4. Finally section 7 concludes.

2 Research Design

2.1 The Derivation of the Empirical Model

Following the seminal papers by Evans (1991) and Schaller and van Norden (2002) and Brooks and Katsaris (2005a,b), we build an empirical framework that accounts for periodically, partially collapsing (positive and negative) speculative bubbles in nominal equity valuations.[4] In such a model, the probability of collapsing is time varying according to the law of motion (DGP),

$$E_t(b_{t+1}) = \begin{cases} \dfrac{(1+r)B_t}{q(|b_t|, b_t^2)} - \dfrac{1-q(|b_t|, b_t^2)}{q(|b_t|, b_t^2)} u(b_t)P_t^a & \textit{with prob.} \quad q(|b_t|, b_t^2) \\ u(b_t)P_t^a & \textit{with prob.} \ 1-q(|b_t|, b_t^2), \end{cases} \tag{1}$$

where b_t is the size of the bubble relative to the actual price, P_t^a, which can, for instance, be expressed as

$$b_t \equiv \frac{B_t}{P_t^a} = \frac{P_t^a - P_t}{P_t^a}.$$

In equation (1), B_t is the total size of the bubble at time t, $u(b_t)$ is a continuous and everywhere differentiable function such that $u(0) = 0$ and $0 \leq \frac{\partial u(b_t)}{\partial b_t} \leq 1$, to be interpreted as the residual value of the bubble after it partially collapses (the second regime in (1)) and $q(|b_t|, b_t^2)$ is the probability of the bubble continuing to exist. Out of plausibility (see Kindleberger, 1996) and earlier empirical evidence (see Hall, Psaradakis, and Sola, 1999)), we assume that $q(|b_t|, b_t^2)$ is a function of the absolute value and/or the square of the relative size of the bubble. Equation (1) conveys the idea that in the surviving regime, returns should be sufficiently high to compensate the investor for the possibility that the bubble may collapse. In fact, extending the work by Schaller and van Norden (2002), in Appendix A, we derive (1) as an extension of a standard present value model (see LeRoy, 1989, for a review of the early literature on present value models

[4] Because in Section 6, we try and convert our model into an operational trading strategy, in the main body of this paper we focus on nominal asset prices and fundamentals.

in asset pricing research). Importantly, when equation (1) is re-expressed in terms of a nominal bubble ($B_t = b_t P_t^a$) and assuming that there are no chances of a collapse ($q(|b_t|, b_t^2) = 1$) then we obtain the formulation of a rational, collapsing bubble in Evans (1991) under constant required rate of return:

$$E_t(B_{t+1}) = (1 + r)B_t,$$

which confirms that the rate of growth (hence, gross return) of a bubble is the same as the one on a stock so that the bubble will be fully rational and (under constant expected returns or risk neutrality) it will satisfy the fundamental Euler condition, as explained in Campbell et al. (1997).

There are two main approaches to measure the fundamental price, that we shall call P_t to distinguish it from the actual asset price, P_t^a. The first one is adapted from Schaller and van Norden's work and is based on a multiple of current dividends, which is the appropriate approach under risk neutrality and when the interest rate is constant and dividends follow a homoskedastic log-random walk process with a drift (i.e., the continuously compounded mean real dividend growth rate) that is constant. The other definition is based on Campbell and Schiller (1987) and allows for the mean dividend growth rate to vary over time.

The early literature, going back to the ground-breaking work by Kindleberger (1996), had reported that the relative size of a bubble is likely to produce an impact on the probability of the very bubble to burst, in the sense that as the (relative) size of a bubble increases, the conditional probability of a collapse (partial or total) occurring might be affected, within sensible boundaries.[5] To capture this mechanism, we use a logit specification to incorporate the size of the bubble into the probability of a bubble to persist,[6]

$$q(|b_t|, b_t^2) = \ell\left(\beta_{q,0} + \beta_{q,babs}|b_t| + \beta_{q,bsqr}b_t^2\right),$$

where ℓ is a standard logistic density function. Note that $\ell\left(\beta_{q,0}\right)$ is the constant probability of being in the bubble surviving regime when the size of the bubble is equal to 0, i.e., the probability of a bubble to find inception when there is not already a bubble in the market. $\beta_{q,babs}$ is the sensitivity of the probability of survival of the bubble to the absolute value of the relative size of the bubble. The absolute value is taken because when $\beta_{q,babs} > 0$, a sufficiently large negative

[5] The functional form of $q(|b_t|, b_t^2)$ should be specified in a way that prevents it from reaching a value of 1 because when $q(|b_t|, b_t^2) = 1$ a rational bubble stops being a periodically collapsing one and is instead expected to last forever. The logit function specified later, in fact, ensures that any unconstrained estimates satisfy $q(|b_t|, b_t^2) \in (0, 1)$.

[6] We follow van Norden and Schaller (1993, 2002) who adopt a probit specification. Compared to the Gaussian probit specification in and Brooks and Katsaris (2005a,b), we find that a logit function guarantees higher numerical stability and appears to be more in line with the literature on regime switching regressions under time-varying probabilities.

bubble may render $q(|b_t|, b_t^2)$ negative as $\beta_{q,0} + \beta_{q,babs}|b_t| + \beta_{q,bsqr}b_t^2 < 0$, which would make no sense. This practice follows Schaller and van Norden (2002).

Following the same steps as in Brooks and Katsaris (2005a, 2005b), this framework for the dynamics of the conditional expectation of the relative bubble can be transformed into a regime switching, state-dependent model for expected returns, in which the driving state is not observable, see Appendix A. Expected asset (stock) returns are given, in each state (S and C), by

$$E_t(r_{t+1}|S) = \left[\mu(1-b_t) + \frac{\mu b_t}{q(|b_t|, b_t^2)} - \frac{1-q(|b_t|, b_t^2)}{q(|b_t|, b_t^2)}u(b_t)\right] \ \ with\, prob.\ q(|b_t|, b_t^2)$$

$$E_t(r_{t+1}|C) = [\mu(1-b_t) + u(b_t)] \qquad with\, prob.\ 1-q(|b_t|, b_t^2), \qquad (2)$$

where S and C are the survival state and the collapsing regime, respectively; r_{t+1}denotes the net return in interval [t, t + 1] and μ is the fundamental net return on the stock, i.e., the rate of return that would prevail under the conditions that (i) there are no bubbles, and hence (ii) the model collapses to a single-state, constant expected return (μ) one as in classical finance (see LeRoy, 1989). Clearly, when the bubble is possible, conditional expected returns are a function of b_t. The interpretation of (2) is that as long as the bubble survives, returns will come from both the fundamental component (net of the relative size of the bubble, $\mu(1-b_t)$) and from the need to compensate an investor for the losses deriving from a potential bubble collapse, which occurs with odds of $1/q(|b_t|, b_t^2)$. Even though the bubbles collapses, such a burst is assumed to be only partial and therefore the compensation for the bubble collapsing needs to be reduced by a factor that depends on how much of the bubble would be left after a partial burst, $u(b_t)$.

We now follow the earlier literature and take a first-order Taylor series approximation of $E_t(r_{t+1}|S)$ and $E_t(r_{t+1}|C)$ in (2) with respect to b_t around a no bubble steady-state value. Dropping the expectation operator E_t, it is straightforward to obtain:

$$r_{t+1}^S = \beta_{s,0} + \beta_{s,b}b_t + \varepsilon_{s,t+1}$$

$$r_{t+1}^C = \beta_{c,0} + \beta_{c,b}b_t + \varepsilon_{c,t+1}$$

$$q(|b_t|, b_t^2) = \ell\left(\beta_{q,0} + \beta_{q,babs}|b_t| + \beta_{q,bsqr}b_t^2\right), \qquad (3)$$

where $\varepsilon_{s,t+1}$ and $\varepsilon_{c,t+1}$ are the unexpected returns at t + 1 in the surviving and collapsing regime, respectively. These shocks are assumed to be independent and identically distributed with a normal distribution with zero mean but possibly regime-specific variance. The model specified in (3) represents our baseline empirical framework.

Brooks and Katsaris (2005a) and Anderson, Brooks, and Katsaris (2010) have augmented the model in (3) by including a measure of abnormal volume, for instance some relative deviation from a recent moving average of dollar volumes. The premise is that some measure of unexpected volume would help with the identification of the bubble regime and hence with the estimation of its empirical persistence. Bubbles have been historically characterised as periods of hectic trading in which investors who "ride" the bubble try to go long in bubbly stocks to re-sell them after a short-period of time, therefore to profit from the upward trend caused by the martingale process appearing in the (rational) bubble (see, e.g., Liao, Peng, and Zhu, 2022). The reference empirical model, then, becomes:

$$E_t(b_{t+1}) = \begin{cases} \dfrac{(1+r)B_t}{q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)} - \dfrac{1-q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)}{q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)} u(b_t)P_t^a & with\ prob.\ q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2) \\ u(b_t)P_t^a & with\ prob.\ 1-q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2), \end{cases} \quad (4)$$

where V_t^x is a measure of abnormal volume in period t and $q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)$ is the probability of the bubble continuing to exist, which is a function of the relative value of the bubble and of the abnormal volume. (4) shows that the expected value of the bubble in the surviving state is a decreasing function of the probability of survival $q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)$. In other words, the greater the probability of collapse, the larger must be the gain on a positive bubble in the surviving state in order to compensate the investor for the possibility of collapse.

Accordingly, the model in (2) can be extended to become:

$$E_t(r_{t+1}|S) = \left[\mu(1-b_t) + \frac{\mu b_t}{q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)} - \frac{1-q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)}{q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)} u(b_t, V_t^x)\right] \quad with\ prob.\ \ q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)$$

$$E_t(r_{t+1}|C) = [\mu(1-b_t) + u(b_t, V_t^x)] \qquad with\ prob.\ \ 1-q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2), \qquad (5)$$

The economic interpretation of (5) is similar to (2), but now both the probability of a collapse and the size of the partial collapse have become a function also of the abnormal volume, for instance:

$$q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2) = \ell\big(\beta_{q,0} + \beta_{q,babs}|b_t| + \beta_{q,bsqr}b_t^2 + \gamma_{q,vabs}|V_t^x| + \gamma_{q,vsqr}(V_t^x)^2\big),$$

where $\gamma_{q,vabs}$ and $\gamma_{q,vsqr}$ are the sensitivities of the probability of survival to the level and square of the measure of abnormal volume. By applying again a first-order Taylor expansion approximation around a zero-bubble value and a corresponding normal dollar trading volume (which therefore implies a zero abnormal volume), we obtain the following empirical switching regression model with time-varying transition probabilities (also indicated as TVPr in what follows):

$$r_{t+1}^S = \beta_{s,0} + \beta_{s,b}b_t + \gamma_{s,v}V_t^x + \varepsilon_{s,t+1}$$

$$r_{t+1}^C = \beta_{c,0} + \beta_{c,b}b_t + \gamma_{c,v}V_t^x + \varepsilon_{c,t+1}$$

$$q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2) = \ell\left(\beta_{q,0} + \beta_{q,babs}|b_t| + \beta_{q,bsqr}b_t^2 + \gamma_{q,vabs}|V_t^x| + \gamma_{q,vsqr}(V_t^x)^2\right), \qquad (6)$$

where $\varepsilon_{s,t+1}$ and $\varepsilon_{c,t+1}$ are the unexpected returns at $t + 1$ in the surviving and collapsing regime, respectively. Within each regime, these shocks are assumed to be independent and identically distributed under a normal distribution with zero mean but possibly regime-specific variance. (6) represents an empirical specification that competes with (3) and that will be tested to derive the best fitting baseline model for our data.

2.2 Estimation Strategy

Models (3) and (6) represent theory-backed extensions of the independent regime switching regression framework described in Goldfeld and Quandt (1976), in which the transition probabilities are time-varying and depend on (a sub-set of) the same exogenous variables that also drive the conditional mean returns.

In order to estimate these models, we use a maximum likelihood approach that exploits both the parametric specifications derived as a result of a mixture of ad-hoc assumptions and first-order Taylor expansions and of the conditionally independent (but possibly heteroskedastic) nature of the shocks. In particular, for instance in the case of model (6), the log-likelihood function is

$$log\mathcal{L}(r_{t+1}|\boldsymbol{\theta}) = \sum_{t+1}^{T} ln\left[P(r_{t+1}|S)\phi\left(\frac{r_{t+1} - \beta_{s,0} - \beta_{s,b}b_t - \gamma_{s,v}V_t^x}{\sigma_s}\right)\right.$$
$$\left.+ P(r_{t+1}|C)\phi\left(\frac{r_{t+1} - \beta_{c,0} - \beta_{c,b}b_t - \gamma_{c,v}V_t^x}{\sigma_c}\right)\right]$$

where $\boldsymbol{\theta}$ is the set of parameters over which to maximise the log-likelihood function, for example, $\boldsymbol{\theta} \equiv [\beta_{s,0}, \beta_{s,b}, \gamma_{s,v}, \beta_{c,0}, \beta_{c,b}, \gamma_{c,v}, \beta_{q,babs}, \beta_{q,bsqr}, \gamma_{q,vabs}, \gamma_{q,vsqr}, \sigma_s, \sigma_c]'$, $\phi(\cdot)$ is the standard normal probability density function, and σ_s and σ_c are the regime-specific standard deviations of the disturbances in the S and C regimes, respectively. Finally, obviously, $P(r_{t+1}|S) \equiv q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)$ and $P(r_{t+1}|C) \equiv 1 - q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)$ and this is usefully replaced inside the expression for the log-likelihood function.

Following Schaller and van Norden (2002), in Appendix B we derive a list of conditions that must hold if the periodically collapsing speculative bubble model has explanatory power for the stock market returns:[7]

[7] Technically, Schaller and van Norden propose a sub-set of these conditions as useful to test whether the data would contain evidence consistent with either the existence of bubbles or "fads", where the latter are defined as conditions of persisting mispricing that however fail to satisfy the martingale conditions proper of bubbles. The subset of conditions for bubbles is: $\beta_{s,0} \neq \beta_{c,0}$, $\beta_{c,b} < 0$ and $\beta_{q,babs} > 0$.

$$\beta_{s,0} \neq \beta_{c,0}$$

$$\beta_{c,b} < 0$$

$$\beta_{c,b} < \beta_{s,b}$$

$$\left.\begin{array}{l} \beta_{q,babs} < 0 \text{ and } \beta_{q,bsqr} = 0 \; or \\ \beta_{q,babs} = 0 \text{ and } \beta_{q,bsqr} < 0 \; or \\ \beta_{q,babs} > 0 \text{ and } \beta_{q,bsqr} < 0 \end{array}\right\} \text{(model globally stable)}$$

$$\gamma_{s,v} > 0$$

$$\gamma_{q,vabs} < 0$$

The first restriction implies that exist two distinct regimes as identified by the general "level" of average stock returns, as opposed to regimes being simply identified by differences in estimated variances ($\sigma_s \neq \sigma_c$ or the sharper $\sigma_s > \sigma_c$). The constraint $\beta_{c,b} < 0$ implies that the expected stock return should be negative in the collapsing regime or at least inferior to the expected stock return in the case a positive bubble persists.[8] The third restriction, $\beta_{s,b} > \beta_{c,b}$, ensures that the bubble yields higher (lower) returns if a positive (negative) bubble is observed in the surviving regime than in the collapsing regime. Of course, the case $\beta_{s,b} > 0 > \beta_{c,b}$ is compatible with the union of the first three constraints. $\beta_{q,babs} < 0$ (with $\beta_{q,bsqr} = 0$) implies that the probability of the bubble continuing to exist is expected to *decrease* as the size of the bubble *increases*, the case emphasised in Brooks and Katsaris (2005a) and Anderson et al. (2010); however, when $\beta_{q,babs} > 0$, the model turns out to be globally well-behaved provided that $\beta_{q,bsqr} < 0$ so that the probability of the bubble persisting becomes concave in b_t and hence eventually decreasing so that, when $b_t \to +\infty$, then (for given level of abnormal volume) $q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2) \to 0$ and the bubble collapses almost surely. The same occurs in the case in which $\beta_{q,babs} \leq 0$ but $\beta_{q,bsqr} < 0$. $\beta_{q,babs} > 0$ (while $\beta_{q,bsqr} = 0$) and $\beta_{q,babs} = 0$ and $\beta_{q,bsqr} > 0$, were advocated as the restriction identifying in bubbles in Schaller and van Norden (2002) and it implies that the probability of the bubble continuing to exist is expected to *increase* as the size of the bubble *increases*.[9] However, even though $\beta_{q,babs} < 0$, the model turns out to be only locally—specifically, for levels of the relative

[8] The opposite holds for negative bubbles: the larger is the negative bubble, the more positive the returns in the collapsing regime. In the following, also driven by the preliminary empirical results obtained in the literature as well as on our data, our comments mostly focus on the case of positive bubbles.

[9] Schaller and van Norden find empirical evidence supporting these conditions in their data. Everything else equal, this may be problematic because in this parametrisation as a positive bubble becomes sufficiently large, then $q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2) \to 1$, the bubble lasts forever almost surely and the model stops being a collapsing bubble one. In any event, in our empirical work, we have failed to find any evidence of $\beta_{q,bsqr} > 0$ holding in our data.

bubble size not exceeding some data-driven thresholds—well-behaved even though $\beta_{q,bsqr} > 0$, so that the probability of the bubble persisting becomes decreasing in b_t; when b_t is not excessive (in a range away from zero) then (for given level of abnormal volume) $q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2)$ decreases and the bubble is more likely to collapse.

Finally, $\gamma_{s,v} > 0$ states that, as volume increases, investors perceive an increase in market risk which shall need to be compensated by a higher risk premium. The restriction $\gamma_{q,vabs} < 0$ indicates that an abnormally high volume signals an imminent collapse of the bubble.

2.3 The Impact of Monetary Policy

Our paper aims to assess whether, how, and to what extent, monetary policy has historically affected the presence of bubbles in aggregate stock valuations in the US. While, it is typical to measure the stance of monetary policy through the Federal funds rate, both to place our focus on the recent period dominated by quantitative easing (henceforth, QE) policies and to seek a single, encompassing indicator of the stance of monetary policy robust to QE strategies that are not directly based on changing the Fed funds rate, we also experiment with measuring monetary policy using Wu and Xia's (2016) shadow rate.[10] We care for three related aspects:

1. Whether monetary policy has impacted the formation and/or the collapsing of bubbles in US real stock prices;
2. If so, how has that occurred, i.e., whether impacting the conditional expectations of the relative bubble size directly (as often claimed in the popular press, by affecting current and expected future discount rates) and/or through effects on the average duration, i.e., the probability of the bubble surviving and further growing over time;
3. Assuming monetary policy has had an impact on the process of bubble formation and growth and that it affects their probability of (at least partially) collapsing, the estimated, quantitative (as opposed to the qualitative) impact of such an effect.

With these goals in mind, we extend the empirical framework developed above to become:

$$r_{t+1}^s = \beta_{s,0} + \beta_{s,b} b_t + \gamma_{s,v} V_t^x + \psi_{s,m} sr_t + \varepsilon_{s,t+1}$$

[10] It is understood that the shadow rate is likely to turn negative to signal a very accommodative monetary policy stance when the Fed intensifies her asset purchase operations to try and conduct expansionary monetary policy at the zero lower bound. Because the shadow rate series provided by Cynthia Wu starts in January 1990 and ends in December 2022, we have backward/forward extrapolated this series to the 1954-1989 and 2023 sample by regressing the shadow rate on the effective Fed funds rate series for the sample 1990-2022 and then using the estimated coefficient values to find the projected shadow rate that would have prevailed given the observed (possibly low but never negative) FFR. Section 3 gives results on the procedure and the estimates obtained.

$$r_{t+1}^{C} = \beta_{c,0} + \beta_{c,b} b_t + \gamma_{c,v} V_t^x + \psi_{c,m} sr_t + \varepsilon_{c,t+1}$$

$$\begin{aligned} &q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2, |sr_t|, sr_t^2) \\ &\quad = \ell\big(\beta_{q,0} + \beta_{q,babs}|b_t| + \beta_{q,bsqr} b_t^2 + \gamma_{q,vabs}|V_t^x| + \gamma_{q,vsqr}(V_t^x)^2 + \psi_{q,mabs}|sr_t| \\ &\quad + \psi_{q,msqr} sr_t^2\big), \end{aligned} \tag{7}$$

where sr_t stands for "short-term rate" (i.e., either the FFR or the projected shadow rate), $\psi_{s,m}$ and $\psi_{c,m}$ are the coefficients that load realised stock returns on the short rate, and $\psi_{q,mabs}$ and $\psi_{q,msqr}$ measure the effect of the short rate on the probability of the bubbles surviving and hence of the probability of collapse. Appendix A provides a heuristic derivation of such an extended model to include the short rate. The model is estimated by ML using the approach outlined in Section 2.2.

While some earlier literature and much popular press (see the Introduction for a few examples) has debated whether $\psi_{s,m} < 0$ so that an expansionary monetary policy would cause higher expected returns and hence, at least on average, contribute to fuel stock bubbles, in our paper we test two alternative hypotheses of interest:[11]

> *H_0-1:* $\psi_{q,mabs} = \psi_{q,msqr} = \psi_{s,m} = 0$, i.e., monetary policy has no impact on stock prices, either directly in the bubble regime or indirectly through the probability of a bubble surviving.[12]

> *H_A-1:* $\psi_{q,mabs} \neq 0 \ \frac{\text{and}}{\text{or}} \ \psi_{q,msqr} \neq 0 \ \frac{\text{and}}{\text{or}} \ \psi_{s,m} \neq 0$, i.e., monetary policy has either a direct impact on expected returns or has an indirect impact through the duration of bubbles, or both.

Under *H_A-1* we can further distinguish four additional sub-cases, that we shall call *H_A-1a*, *H_A-1b*, *H_A-1c,* and *H_A-1d*:[13]

> *H_A-1a:* $\psi_{q,mabs} < 0, \ \psi_{q,msqr} = 0, \ \frac{\text{and}}{\text{or}} \ \psi_{s,m} < 0$, i.e., expansionary monetary policy *inflates* bubbles directly and makes them likely to survive *longer* (hence allows them to inflate *more*) i.e., also indirectly. In this case, at least qualitatively, easy monetary policies may carry a heavy burden of responsibility for the historical record of US stock bubbles. Of course, the converse is that monetary tightening deflates bubbles directly and makes

[11] In what follows, *H_0–n* refers to a null hypothesis and *H_A–n* to the corresponding alternative hypothesis that we find useful to describe for its economic implications.

[12] The model mostly implies predictions on the impact of monetary policy in the bubble regime but it is realistic to also expect that $\psi_{c,m} = 0$ holds.

[13] Needless to say, the following sub-cases fail to exhaust the domain of possibilities, for instance $\psi_{q,mabs} > 0, \psi_{q,msqr} > 0$ or $\psi_{q,msqr} < 0, \ \psi_{q,mabs} < 0$ fail to be featured in our list. The former case appears problematic because a sufficiently high short rate would then, ceteris paribus, lead to $q\big(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2, |sr_t|, sr_t^2\big) \to 1$ and to a bubble persisting forever; the latter case is instead possible and simply illustrates an extremely strong and convex impact of monetary policy on bubbles.

them likely to burst earlier (hence allows them to inflate over shorter periods).

H_A-1b: $\psi_{q,mabs} > 0,\ \psi_{q,msqr} = 0, \underset{\text{or}}{\overset{\text{and}}{-}}\ \psi_{s,m} > 0$, i.e., expansionary monetary policy *deflates* bubbles directly and makes them *less* likely to survive (hence allows them to inflate *less*). Importantly, under this alternative hypothesis, monetary policy does not necessarily actively fight bubbles, because of the contrasting effects it yields. A monetary tightening would instead inflate bubbles directly but make them less likely to survive (hence allows them to inflate less).

H_A-1c: $\psi_{q,mabs} > 0,\ \psi_{q,msqr} < 0$ which implies a concave relationship between $q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2, |sr_t|, sr_t^2)$ and the short rate and $\psi_{s,m} < 0$, so that the short rate eventually (as it becomes large, say in absolute value to encompass the case of the shadow rate) increases the probability of a bubble to burst, while also exercising a direct, moderating effect on the conditional mean returns as long as the bubble persists. Of course, a monetary expansion of intermediate strength would increase the chances of a bubble to last even though eventually (as the short-rate declines towards zero) such an effect may be moderate, while also exercising a direct, triggering effect on the conditional mean returns as long as the bubble persists.

H_A-1d: $\psi_{q,mabs} < 0,\ \psi_{q,msqr} > 0$ which implies a convex but non-monotonic relationship between $q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2, |sr_t|, sr_t^2)$ and the short rate and $\psi_{s,m} < 0$. In this case, the short rate—specifically, for levels of the short rate near zero, on both sides of it in the case of the shadow rate—exercises a well-behaved effect on the probability of the bubble to burst;[14] because $\psi_{s,m} < 0$, the short rate also exercises a direct effect on the conditional mean returns. Yet, in this case, $\psi_{q,mabs} < 0,\ \psi_{q,msqr} > 0$ are globally non-viable as when the short rate grows, the bubble is made infinitely persistent.

2.4 Measuring Deviations from Fundamental Prices

There are two different approaches proposed in the earlier literature and concerning the measurement of b_t, i.e., the size of the bubble relative to the actual price. The first method follows Brooks and Katsaris (2005a,b) and takes steps from the classical Gordon's (1962) present value formula,

[14] This means that the probability of the bubble persisting is decreasing in sr_t so that when sr_t is in a range away from zero but also within given threshold (for given level of abnormal volume and relative bubble dimension), $q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2, |sr_t|, sr_t^2)$ decreases and the bubble is more likely to collapse.

$$P_t = \frac{D_t}{r - g} \tag{8}$$

where g is the dividend growth rate. Under the assumption that the stock market prices obey the following period-to-period (risk-neutral) arbitrage condition,

$$P_t = \frac{1}{1 + r} E_t(P_{t+1} + D_{t+1}),$$

in which r is the required rate of return (supposed to be constant and equal to the risk-free interest rate) and that the log-dividends ($d_t \equiv lnD_t$) follow a random walk with a drift

$$d_t = \alpha + d_{t-1} + \sigma_d \epsilon_t$$

the fundamental price is, then, shown to be a simple multiple of current dividends,

$$P_t = \rho D_t$$

where

$$\rho = \left[\frac{1 + r}{e^{\left(\alpha + \frac{\sigma_d^2}{2}\right)}} - 1 \right]^{-1}$$

The proportional deviation of actual price (P_t^a) from the fundamental one (P_t) is:

$$b_t = \frac{P_t^a - P_t}{P_t^a} = 1 - \rho \frac{D_t}{P_t^a},$$

in which ρ is approximated with the sample mean of the price dividend ratio $\overline{P^a / D}$ so that the relative bubble b_t can be alternative expressed as:

$$b_t = 1 - \frac{\overline{P^a / D}}{P_t^a / D_t}, \tag{9}$$

i.e., one minus the ratio between the historical mean price-dividend ratio and the time t price-dividend ratio so that a currently above average price-dividend ratio becomes the source of a positive bubble estimate.

A second method has been proposed by Campbell and Shiller (1989) and adapted to estimate the size of bubbles by Schaller and van Norden (2002). Because this method implies a more tightly parameterised model to provide guidance to extract estimates of b_t, it is used to check the robustness of our main result to equation (9) and it is presented in Section 5.5.

3 The Data

Our baseline sample of data covers the period June 1954 – December 2023, for a total of 836 observations. The start date of the study is determined by the availability of data on (the existence of) the effective Fed funds rate. However, the 67-year long monthly sample appears to be long enough to cover a number of (alleged) bubbles, followed by stock market crashes and recoveries, for instance, the two oil shocks in the 1970s, the 1986-1988 market surge ended with the spectacular crash of October 1987, the dot.com and the sub-prime real estate bubbles that have occurred at the turn of the millennium and then, at least allegedly, between 2005 and 2008. As mentioned in the Introduction, also recent media commentary has discussed the probability of a stock market bubble inflating after 2021.

The data concerning stock prices, dividends and earnings and (where needed) the 1-month Treasury bill risk-free rate are kindly made available by Bob Shiller through his personal web site.[15] Prices, dividends and earnings all concern the Standard & Poor's composite index. The data on the effective Fed fund rates are instead sourced from FRED II® from the Federal Reserve Bank of St. Louis.

The abnormal dollar volume V_t^x is computed as the monthly percentage difference between the dollar volume concerning the stocks in S&P index and a 12-month moving average built starting from the same monthly series.[16] In a few specifications, we try an alternative proxy for the rational "trading exuberance" (see Shiller, 2015) that is allegedly typical of bubbles and resort to Baker and Wurgler's (2006, 2007) orthogonalized sentiment index. In particular, BW's index depends on trading volume (the turnover ratio), the number of initial public offerings (IPOs) and the average first-day returns on IPOs, the proportion of equity issues relative to the total equity and debt issues, the average discount of closed-end fund prices relative to their net asset values, the dividend premium (the difference in market-to-book ratios between dividend-paying and non-dividend-paying stocks) and the net inflows into mutual funds.[17] In Section 5.4 of the robustness tests we also experiment with raw, non-orthogonalized version of BW's sentiment indicator.

The shadow Fed fund rate monthly series for the United States is taken from Cynthia Wu's web site.[18] Because the shadow rate has been formally estimated by Wu and her co-authors only

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[15] See http://www.econ.yale.edu/~shiller/data.htm.

[16] In Section 5.3 of the robustness checks, we have also experimented with a 6-month moving average of volume.

[17] The orthogonalized version of BW's index is designed to isolate pure sentiment effects by removing influences from fundamental economic variables, i.e., it is adjusted to be uncorrelated with macroeconomic variables such as industrial production, employment, inflation and short-term rates. Moreover, note that the notion of volume employed by BW when constructing their index is different from our notion or abnormal, relative (percentage) monthly volume, even though the two variables (hence, to some extent also BW's index) are expected to be positively correlated.

[18] See https://sites.google.com/view/jingcynthiawu/shadow-rates.

starting in January 1990 (due to the availability of interest rate options data), we have regressed the available January 1990 – February 2022 shadow Fed fund rate on the effective Fed fund rate, obtaining the fitted values (standard errors are in parentheses):

$$\widehat{sr}_t = \underset{(0.040)}{0.062} + \underset{(0.010)}{0.958} \times ffr_t \qquad (R^2 = 0.972)$$

We use this estimated model to compute the implicit shadow rate also with reference to the June 1954 – December 1989 and then March 2022 – December 2023 periods, even though our finding is that over these sub-samples the correlation between the fitted shadow rate (which is always positive) and the effective Fed fund rate is practically 1 (0.989).

Table 1 reports the basic summary statistics for our data set. The S&P 500 (excess) returns show statistics in line with common knowledge on the behaviour of aggregate stock returns in the United States. Their mean is 11% (7% in the case of excess returns) in annualised terms, with a median of 14.6% (10.6%) that considerably exceeds the mean, an indication of substantial left-skewness (-0.90). Both net and excess returns display kurtosis in excess of a standard Gaussian distribution (6.6). The stock return standard deviation (e.g., 3.5% per month) implies an annualised volatility of 12.2% which corresponds to the low end of the typical, historical values but that matches some recent empirical evidence (see, e.g., Cooper and Maio, 2019). The S&P index is also characterised by relatively high price-dividend ratios (on average of almost 34), even though peaks in excess of 40 have been reached on a few occasions. In fact, the price-dividend ratio is marked by a substantial volatility of almost 9 and by positive skewness, as one would expect given that this ratio can only be positive.

The statistics concerning the short-term rates driven by monetary policy are all rather similar, with the exception of those concerning the shadow rate, because the latter covers a much shorter sample period in which this option-implied policy rate measure is explicitly allowed to become negative (as much as -0.25% per month) to capture the expansionary effects of quantitative easing policies. For instance, the FFR is characterised by annualised mean and median of 4.6% and 4.2%, respectively, and by an annualised volatility of just 1% per year, which is to be expected. The short-term rates are also characterised by positive excess kurtosis and substantially positive skewness, as one would expect of nominal rates subject to a zero lower bound. Because it just spans less than half of the overall sample and this period was characterised by low rates, the shadow rate series reports much lower mean and median rates (2.3 and 2.1%, respectively, in annualised terms) than FFR or 1-month T-bill rates, and zero skewness as a result of the fact that the zero lower bound is essentially removed in this case.

Finally, Figure 1 presents the estimate of the relative bubble derived from equation (9) applied to the data above (the boldfaced blue curve, measured on the left scale) plotted along with the monthly FFR (measured on the right scale). The figure also displays the estimate of the relative bubble obtained from a Campbell and Shiller's (1987) method to be used in Section 5.5 as a robustness check (the dashed red curve). The latter method allows predictability patterns in stock and dividend changes to be captured. Visibly, the two measures of b_t are strongly, positively correlated, the only limited weakening is occurring between 1978 and 1982. The bubble is almost always positive, with the notable exception of the late 1970s and early 1980s, where it turns negative and relatively large, exceeding in a few years 50% of the overall fundamental evaluation. The only other episode of a persistent under-valuation (arguably, of $b_t < 0$) occurs in occasion of the Great Financial Crisis, in 2008-2009. The Covid-19 pandemic shock appears as a transitory two-month dip occurred in February and March of 2020. A massive (also in this case, exceeding by 50% of the fundamental valuation between 1999 and 2001) bubble that has lasted more than a decade occurred between 1993 and 2006, even though it started deflating after 2002. A simple, bird's eye comparison between the dynamics of both estimates of b_t and the FFR, alerts us of the fact that there seems to exist a clear negative correlation between the FFR and the relative size of the S&P bubble. The empirical question is whether and how simple models of correlations, such as linear predictive regressions may suffice to capture such a relationship.

4 Main Empirical Results

4.1 Benchmark Models

Table 2 reports full sample estimates of a range of models that include six benchmark models. A benchmark model consists of a restricted version of the general empirical model of periodically collapsing, rational bubbles presented in Section 2. Models 1 – 3 in Table 2 are benchmarks because they are single-state, linear regression models that exclude the occurrence of regimes. This means that there are no collapsing bubbles and instead these go on forever:

$$r_{t+1} = \beta_0 + \beta_b b_t + \gamma_v V_t^x + \psi_m sr_t + \varepsilon_{t+1} \tag{10}$$

where $q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2, |sr_t|, sr_t^2) = 1$. Of course, if $\beta_b \gg 0$ bubbles acquire a self-sustaining nature, as bigger bubbles increase stock returns and push prices higher, even though the model in (10) is not an autoregressive one in a technical sense and the definition of relative bubbles in equation (9) involves dividends, that are instead not featured in model (10). The opposite occurs when $\beta_b < 0$ as large relative bubbles depress expected returns and hence limit the extent of subsequent bubble dynamics. Importantly, ψ_m captures the impact of the short-term rate on aggregate stock returns, even though in this case of a standard linear regression, the sign of ψ_m can be hardly mapped into a claim on the effects of monetary policies on bubbles. Model 1 in Table 2 consists of a restricted version of the regression in (10) in which $\psi_m = 0$. Model 3 is a version of the regression (10) extended to Baker and Wurgler's sentiment index. Note that across models 1 – 3, the variance of the regression residuals is assumed to be constant over time and hence the model is homoscedastic.

Table 1 shows that these linear benchmarks provide rather a poor fit to S&P index returns. The Hannan-Quinn's information criteria (henceforth, HQIC; particularly suitable to model selection applied to non-linear frameworks, see, e.g., Psaradakis et al., 2009) turns out to be far above those achieved by the remaining models, including those offered by simple regime switching model in which the constant probability q is an estimable constant. For instance, the best among these three regressions, i.e., model 1, scores a H-Q criterion of 5.356 which is massively above the 5.136 achieved by the heteroskedastic switching regressions 4 and 5 and above the 5.165 of the homoscedastic model 6. Moreover, in both models 2 and 3, the ML estimates of ψ_m fail to be statically significant and the same occurs to abnormal volume when the regression is extended to also include sentiment. Interestingly though, in all models $\hat{\beta}_b$ turns out to be positive and statistically significant.[19]

A further set of benchmarks, but of a two-state, regime switching type, are represented by models 4 – 6 in Table 1. For instance, in the case of model (5), we have:

$$r_{t+1}^{S} = \beta_{s,0} + \beta_{s,b} b_t + \gamma_{s,v} V_t^x + \psi_{s,m} sr_t + \varepsilon_{s,t+1}$$

$$r_{t+1}^{C} = \beta_{c,0} + \beta_{c,b} b_t + \gamma_{c,v} V_t^x + \psi_{c,m} sr_t + \varepsilon_{c,t+1}$$

$$q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2, |sr_t|, sr_t^2) = \ell(q_0) \qquad (11)$$

where $\varepsilon_{s,t+1}$ and $\varepsilon_{c,t+1}$ are the unexpected returns at t + 1 in the surviving and collapsing regime, respectively. Within each regime, these shocks are assumed to be independent and identically distributed under a normal distribution with zero mean but possibly regime-specific variance. In this case, the rational bubble may collapse with a constant probability that depends on q_0. This means that even though $\beta_b < 0$ implies that large relative bubbles depress expected returns and hence limit the extent of subsequent bubble dynamics, the very probability of the bubble fails to depend on the relative size of the bubble. In particular, model 4 corresponds to a version of 5 that fails to include the FFR in both regimes (i.e., $\psi_{s,m}$ = $\psi_{c,m}$ = 0). Model 6 is instead a restricted version of 5 characterised by the same regime-specific conditional mean functions but in which the standard errors of the shocks in the two regimes are restricted to be the same, i.e., such that the model is homoskedastic.

[19] Everywhere in this paper, standard errors (hence, the corresponding p-values) have been computed by inverting the final Hessian matrix and the numerical optimisation has been performed using the BFGS algorithm.

Table 1 shows that models 4 – 6 all mark a significant improvement over the simple regressions in 1 – 3 in terms of trading off a better fit to data vs. an increase in the number of estimated parameters, which grow from 4-6 in the case of the regressions 1 – 3 to 9-11 in the case of models 4 – 6. In particular, model 5 shows that the relative size of the bubble has self-sustaining effects as $\hat{\beta}_{s,b} = 6.47$ and it is highly statistically significant in the bubble regime while FFR reduces stock returns in both regimes but this effect is statistically significant only in the collapse regime (i.e., $\hat{\psi}_{c,m} = -0.97$ is significant at a 5% size). As one may expect, the bubble regime is significantly more volatile than the collapse regime, with annualised volatilities of 13.1 and 8.3 percent, respectively, before accounting for the volatility-inducing effects of regimes, of course.[20] However, model 5 shows a constant probability of 0.222 of the system to be in a bubble state and hence a probability of 0.778 that the bubble may collapse on every month, assuming it existed to start from. The resulting durations are therefore modest (1.3 and 4.5 months, respectively) which make this model hard to refer to as a solid characterisation of stock market dynamics. Finally, model 6 loses considerable power to fit the data because of the homoskedasticity restriction: for instance, its H-Q criterion climbs up from 5.136 of model 5 to 5.165. Also because of this evidence, the remaining models in Table 1 all entertain regime-specific volatilities.

Models 7 and 8 in Table 1 concern regime switching models in which the regime switching transition probabilities are time-varying and turn out to be function of the relative bubble size and of abnormal percentage volume. In fact, model 8 differs from 7 because it replaces abnormal volume in the latter with the sentiment index.[21] Despite the fact that the number of parameters climbs up from 9 to 13, for instance when going from model 4 to 7 in Table 1, the H-Q criterion declines (from 5.136 to 5.130) which bears witness to the fact that the additional four parameters specified in the logistic function pinning down transition probabilities help the fit and fully compensate the expected cost in terms of parameter uncertainty. Of course, this improvement measured through an (admittedly) marginal decline in the information criterion shall need to be validated in terms of OOS forecasting accuracy and in terms of realised portfolio performance, which are comparative features to be measured in Section 6. For instance, model 7 implies an (indirect) self-exciting effects in the bubble regime because $\hat{\beta}_{s,b} = 5.94$ and is statistically significant, while $\hat{\beta}_{c,b} = -0.05$ and insignificant in the collapsing regime. However, because $\hat{\beta}_{q,babs} = 11.36$ (with almost a zero p-value) but $\beta_{q,bsqr} = -22.58$ (and highly statistically

[20] Crucially, these are estimates of the volatilities of regime-specific regression residuals, i.e., over and above the volatility of S&P returns that is captured by the predictor variables included on the right-hand side of the conditional mean functions.
[21] However, while comparing the performance of the two models, we should keep in mind that the available time series for the sentiment index are much shorter vs. abnormal stock trading volumes.

significant), we have the typical situation in which relative bubbles are initially, locally self-sustaining, as small bubbles increase the probability of their survival, but eventually as b_t grows larger and larger, the square term will dominate and force the bubble to collapse, as $\ell(\beta_{q,0} + \beta_{q,babs}|b_t| + \beta_{q,bsqr}b_t^2 + \gamma_{q,vabs}|V_t^x| + \gamma_{q,vsqr}(V_t^x)^2) \to 0$. Interestingly, the abnormal percentage volume never plays a dominant role, as both $\gamma_{s,v}$ and $\gamma_{c,v}$ are estimated to be small (with $\hat{\gamma}_{s,v}$ oddly negative) but fail to impact in any accurately measurable way the regime transition probabilities (i.e., $\gamma_{q,vabs}$ and $\gamma_{q,vsqr}$ are small and never significant). Figure 3 Panel (a) (to be extensively commented later) also shows the filtered, real-time probabilities of the bear regime implied by the ML estimation of model 7. The model shows rather steady bubble probabilities that oscillate between 0 and 0.3 between 1955 and 1995, with a spike at 0.4 in 1974 (which is sometimes recognised as the end of the 'Nifty Fifty' period). Starting in 1996, such probabilities climb up to a new range (0.2, 0.6) indicating a structurally higher real-time perceived probability of the US stock market being in a bubble, with spikes corresponding to the periods 2000-2001 (the alleged "dot com" bubble), 2007-2008 (the real estate, credit bubble), and as of recently the post-covid bubble in 2021-2022.

4.2 Main Models Including the Federal Funds Rate

We now move on to describe the main empirical results of our paper. Models 9 through 12 in Table 2 are all regime switching models in which the transition probabilities are time-varying and allowed to depend (within the logistic function specification $\ell(\cdot)$ in equation (7)) on the (absolute values and squares) of the relative bubble size, of abnormal percentage volume and of the short-term rate capturing monetary policy (denoted as m). While model 9 already marks a small improvement vs. model 7 in terms of its H-Q information criterion, with its 17 parameters the model happens to be insufficiently parsimonious. In fact, both $\hat{\gamma}_{q,vabs}$ and $\hat{\gamma}_{q,vsqr}$ appear to have been imprecisely estimated. This explains why the best trade-off between in-sample fit and expected predictive performance is instead achieved by model 10, which features 15 parameters and essentially removes the two variables that depend on abnormal percentage volume from the specification of the logistic function $\ell(\cdot)$. Indeed, the H-Q criterion declines from 5.128 in the case of model 9 to 5.122 in the case of model 10.

Model 10 represents the best specification achieved in this paper and proposes several features of interest. The relative bubble size variable b_t keeps playing the expected role: $\hat{\beta}_{s,b} = 5.43$ and precisely estimated in the bubble regime, while $\hat{\beta}_{c,b} = -0.36$ not statistically significant, indicate that bubbles are indirectly self-sustaining when they are inflating and then—also because they become small and modestly volatile—they stop affecting returns upon collapsing. Most crucially however, the comparison between model 10 and model 7 illustrates the distinctive role played by monetary policy in inflating rational bubbles and in causing their collapse. In model 10, a higher FFR deflates and contrasts bubbles through a direct channel, that emerges from the conditional mean estimates. Interestingly, this occurs both in the bubble (when $\psi_{s,m} = -3.59$ with a p-value of 0.028) and in the collapsing ($\psi_{c,m} = -0.92$ with a p-value of 0.043) regimes, even though a one-sided Wald test of the null of $\psi_{c,m} > \psi_{s,m}$ rejects the null with a p-value of 0.063, which is consistent with a restrictive monetary policy taming bubbles more aggressively when they are inflating vs. when they are already in a collapsing state. However, monetary policy also exercises a complex, non-linear effect—through its impact on the regime transition probabilities—on the probability of a bubble bursting and hence on its expected duration.[22] The economic significance of the impact of the FFR on stock returns is of first-order: one standard deviation shock to increase the FFR (0.30% on a monthly basis, assuming FFR changes are completely unpredictable) decreases expected returns by 1.08% in the bubble regime and by 0.28% in the collapse regime. Therefore, it is certainly not through a conditional mean return channel that progressive reductions in the short rate may have caused high realised mean stock returns and bubbles in our sample.[23] In fact, in model 10, an expansionary policy that lowers the FFR would inflate a bubble through a direct channel, that emerges from the conditional mean estimates. Also in this case, monetary policy also has a non-linear effect that goes through the regime transition probabilities on the expected duration of a bubble.

One recurring, specific property which makes the ex-post identification (interpretation) of the economic nature of the regime easier is that the bubble regime tends to carry an estimated, annualised volatility that is larger (in the case of model 10, almost 49% higher) than in the collapsing regime, even though the collapse may include drastic corrections in equity valuations that may increase ex-post realised variance. The intuition is that in the bubble regime, equity prices are not aligned with fundamentals and this creates unstable opportunities for speculative investing that tend to plausibly manifest themselves along with higher realised volatility.

Yet, the most interesting effects of both relative bubbles and of the short-term rate occur with reference to their effects on the transition probabilities between the two regimes. For both features, we obtain that the coefficients associated to their absolute values ($\hat{\beta}_{q,babs}$ and $\hat{\psi}_{q,mabs}$) are precisely estimated to be positive and large (the coefficients are 9.08 and 4.50, respectively);

[22] As in earlier commentary concerning model 7, $\gamma_{s,v}$ and $\gamma_{c,v}$ are estimated to be small (with $\hat{\gamma}_{s,v}$ oddly negative).

[23] Little changes if the one standard deviation shock to the FFR is made conditional on the bubble collapsing regime of the S&P index, as these are identical and equal to approximately 0.30% per month.

the coefficients associated to their squares ($\hat{\beta}_{q,bsqr}$ and $\hat{\psi}_{q,msqr}$) are precisely estimated to be negative and large (the coefficients are -19.37 and -2.22, respectively). Figure 2 displays the behaviour of the probability of a bubble to survive either as a function of the short-term rate (for a fixed relative bubble size) in Panel (a) and of the relative bubble size (for a given short-term rate) in Panel (b), respectively, and as a function of both the relative bubble size and of the short-term rate in the three-dimensional plots at the bottom.[24] In Panels (a) and (b) is visible that both sr_t and b_t produce a non-monotonic effect that is symmetric around a zero bubble in the case of b_t, while it reaches its maximum for a FFR of approximately 1% per year and for relative bubbles sizes of slightly less than 30% in absolute values. However, because of the negatively signed quadratic terms appearing in the logistic probability function, the marginal effect of FFR makes $q(|b_t|, b_t^2, |sr_t|, sr_t^2)$ essentially zero so that a bubble is mechanically bursting for a FFR of approximately 3 percent per year and for relative bubbles sizes in excess of just less than 100%. Of course, while short-term policy rates exceeding 3 percent appear to be rather normal and easy to engineer, in the presence of easy monetary policy, a relative bubble size of 50-100% of the fundamental value appears to be rather massive in historical perspective. Panels (c) and (d) jointly visualise these features, showing that the probability of bubbles inflating mount and easily exceed 80% for interest rates in the range of 1-2 percent per year and relative bubbles size between 30 and 50%.

Panel (d) of Figure 2 is of high significance because it stresses that however, when interest rates are in the 1-2 percent range, even in the absence of a current bubble, the probability of a bubble surviving exceeds 50% on every single month: easy FFR policies are consistent with the formation of bubbles even when these are initially absent. Nonetheless, Panel (c) reveals that for short-term rates higher than 3 percent, such a probability declines rapidly and this is the case especially when bubbles in the relative size range between 50 and 100% exist to start with. In fact, for short-term rates exceeding 3.5 percent per year, model 10 implies the bursting of all bubbles, irrespective of their initial size or existence. Although it is very different in its theoretical underpinnings (here, we formally start from the notion of rational, periodically collapsing bubbles) and especially in its empirical formulation (here, a regime switching heteroskedastic TVPr model), the estimates of model 10 are qualitatively consistent with the main findings of Galí and Gambetti (2014): there is a range of (relatively low) FFR rates for which bubbles are supported by the FFR being increased; yet, eventually, as the FFR is increased aggressively, it will end up taming the bubble itself. Yet, in our framework, such an effect does not occur through time

[24] The lower plot to the right (Panel (d)) is identical to the lower left (Panel (c)) plot but rotated to show the probability of a bubble surviving for low short-term rates, with emphasis on the range 0-2 percent.

variation in the coefficients of an encompassing VAR model, but through the non-linearities of the dependence of the logistic transition probabilities from the FFR itself.[25]

Figure 3 compares 1-year moving averages of real-time filtered probabilities of a bubble regime across alternative models using model 10 as a benchmark, given its superior in-sample fit penalised by a measure of model's complexity, as captured by the Hannan-Quinn criterion (HQIC). In the plots, we display 12-month moving averages of filtered probabilities to capture in real-time a smooth measure of the evolving perception by an investor to be in a rational bubble regime. Panel (b) visualises the effects of making the bubble filtered probabilities a function of explanatory variables, such as the relative size of the bubble and the short-term rate by comparing the probabilities inferred from models 10 and 5, respectively. The dashed, red and larger font curve referring to model 5 illustrates the ability of a regime switching regression to capture the alternating perceptions of a bubble conditions to prevail and in particular are visible two features. First, while between 1955 and 1995 such probabilities had steadily oscillated between 0 and 0.3, between 1996 and 1997 a jump occurs and the moving average of the probabilities rises to 0.1-0.5 with occasional spikes in excess of 0.6. Second, model 5 delivers probabilities that spike and then decline in correspondence to the end of four major (alleged) bubble episodes (the Fall of 1987, the Summer of 2002, the Summer of 2008 and the Fall of 2022). Clearly, the bubble probabilities fail to be affected by the FFR, as this does not enter the logistic probability specification in model 5. Even in the perspective of Figure 3, model 10, by including the FFR in the logistic specification of the bubble probabilities, marks a meaningful improvement over model 5 in two ways: while before 1995 the likelihood of bubbles tends to be lower and (especially between 1977 and 1986) is very close to zero, the increase from the late 1990s is more remarkable; major, alleged bubble episodes (such as 1998-2001, 2007-2008 and 2010-2015) are characterised by relative high moving averages of bubble regime probabilities, reaching level just short of 0.7 in correspondence to the real estate/sub-prime bubble.

Table 2 provides evidence on the estimation results for two additional models, numbered 11 and 12. Model 11 is identical to model 10 but adds the BW's sentiment index to the conditional mean and logistic transition probabilities equations, thus featuring a rather large number of estimable parameters (21) using many less observations than models 7 or 10. This means that when we move from model 10 to 11, we go from a rather reassuring saturation ratio of 55 to 33 at best. We obtain that the sentiment index is hardly ever significant and—when one were to fix the sample of the estimation of models 11 and 10 to be comparable, i.e., limited to July 1965 – June

[25] As shown in Table 2, the conditional mean of S&P returns is always lowered by higher values of the FFR. Note that the existence of a non-linearity in the formula for the logistic transition probabilities features a sort of non-linearity nested within some over-arching non-linearity.

2022—the corresponding HQIC climbs up from 5.164 to the 5.170 as shown in Table 2. As a result, more interest ought to be paid to model 12 which includes once more the BW's sentiment index but only in the conditional mean function, where in the bubble state higher sentiment predicts lower S&P index returns (this is a standard finding, see, e.g., Baker et al., 2012). Interestingly, the inclusion of BW sentiment in both models 11 and 12 does not seem to replace or make abnormal volume insignificant in the switching regression specifications. Therefore, model 12 is characterised by a HQIC of 5.160 that slightly improves over the 5.164 scored by model 10 on the BW sentiment-constrained sample.

Panel (d) of Figure 3 closes this Section by reporting in the same plot the 1-year moving average of the filtered probabilities of a bubble derived from model 10, the FFR and of the expected (predicted) S&P returns derived from the model (measured on the right scale). Visibly, expected returns tend to be positive and stable when bubble probabilities are low. On the contrary, when filtered bubble probabilities are persistently high to the point of raising the corresponding 1-year moving average, expected returns are more volatile and may therefore turn exceptionally low (as in the case of 2008-2009). Finally, given the prominent role of the squared value of the FFR (multiplied by a negative coefficient) in the logistic probability specification for model 10, in the plot low (high) FFR tends to occur along with rising and relatively high (low) probability of a bubble (with a correlation of -0.6), which is also consistent with Figure 2.

To close this Section, it seems appropriate to formally assess the hypotheses that we have written down and specified earlier in the light of the ML estimates of models 10 and 12 presented in Table 2. Clearly H_0-1 can be rejected as $\psi_{s,m}(= \psi_{c,m}) = \psi_{q,mabs} = \psi_{q,msqr} = 0$ can be rejected using standard 5 percent sized tests, so that monetary policy yields an accurately measured impact on stock prices, both directly in the bubble regime and indirectly through the probability of a bubble surviving.[26] These conclusions derive from both models 10 and 12 and appear robust to including sentiment as a predictor and/or to using a different (shorter) estimation sample. Furthermore, we are in a position to discriminate among the four additional sub-cases isolated with reference to H_A-1. Clearly, in the case of both models, we obtain empirical evidence consistent with H_A-1c: $\psi_{s,m} < 0$, an expansionary monetary policy inflates bubbles directly; $\psi_{q,mabs} > 0$ but $\psi_{q,msqr} < 0$, which implies a concave relationship between $q(|b_t|, b_t^2, |V_t^x|, (V_t^x)^2, |sr_t|, sr_t^2)$ and the short rate, so that the short rate eventually (as it becomes large, say in absolute value to encompass the case of the shadow rate) increases the probability of a bubble to burst, ensuring global stability of the dynamic model.

[26] In model 12, including orthogonal sentiment as an explanatory variable but estimated on a shorter sample, $\psi_{c,m}$ turns out to be imprecisely estimated but this coefficient was not included in the formal statement of H_0-1.

4.3 Model Specification Tests

Similarly to Brooks and Katsaris (2005a) and as it is typical of the applied econometrics literature, besides resorting to a sorting of the models based on the H-Q information criterion, we also proceed to apply likelihood ratio tests (henceforth, LRT) to assess whether restricted versions of our baseline (in particular, model 5) and key (in particular, model 10) models might be simplified and paired down to exclude either predictors and/or non-linear effects in the time-varying transition probability function, yet preserving a not significantly worse fit to the data.

In Table 3, starting from model 5 which is simply pasted from Table 2, we test a model in which the intercept β_0 is restricted to be the same across regimes, thus indicating that the regimes can be distinguished through their different volatilities and predictability patterns but not because these carry a different, structural mean level. Under this simple restriction, the number of parameters declines from 11 to 10 and the maximised log-likelihood from -2107.796 to -2112.089, and this decline by 4.293 log-likelihood points (that multiplied by two = 8.586) is highly statistically significant under a Chi-square distribution with one degree of freedom (from 11-10 = 1), as the associated p-value is 0.003.[27] This means that it is important, already in the baseline model with constant transition probabilities, that $\beta_{s,0}$ and $\beta_{c,0}$ be separately estimable. Moving towards the right in Table 3 (as one does, the Reader will note that the number of estimable parameters tends to decline as restrictions of growing strength are imposed), we test the so-called "fads" model by Cutler et al. (1991) in which returns are predictable, although mean returns do not differ across regimes, i.e., $\beta_{s,0} = \beta_{c,0}$, $\beta_{s,b} = \beta_{c,b}$, $\psi_{s,m} = \psi_{c,m}$. Furthermore, the deviation of actual prices from the fundamentals has no predictive ability for the probability of being in the surviving regime, $\beta_{q,babs} = \beta_{q,bsqr} = \psi_{q,mabs} = \psi_{q,msqr} = 0$. The returns in the two regimes are characterised by different variances of residuals (σ_s and σ_c) but are the same the linear functions of bubble deviations. We find that twice the log-likelihood difference equals 130.47 (= 2 x (-2107.796 + 2173.030)) which, under a Chi-square distribution with 5 degrees of freedom (as the number of parameters declines from 11 to 6 when one goes from model 5 to the fads version of the model), implies a rejection of the null of no difference in fit between the two models, with a p-value of 0.000 (similarly to Schaller and van Norden, 2002). As a final step, we also test model 5 against an even simpler two-state Gaussian mixture model specification in which there is no predictability from either monetary policy or the relative size of existing bubbles for stock returns and only the mean ($\beta_{s,0}$ and $\beta_{c,0}$) and the variance intercepts (σ_s and

[27] The Chi-square distribution for the (log-) likelihood ratio test holds only asymptotically or under the assumption of correct specification of the Gaussian shocks characterising the regime switching specification. In the former perspective, our sample of 829 observations turns out to be an important strength.

σ_c) are allowed to switch, but with constant, independent probabilities. Once more, we find that twice the log-likelihood difference equals 123.47 (= 2 x (-2107.796 + 2169.532)) which, under a Chi-square distribution with 6 degrees of freedom (as the number of parameters declines from 11 to 5 when go from model 5 to mixture model), implies a rejection of the null of no difference in fit with a p-value of 0.000. All in all, Table 3 shows that both in terms of minimisation of the HQICs and when LRT-type tests are performed, a full model that includes both predictability and heteroskedastic regimes is needed by our S&P return data over the relevant sample.

The rightmost portion of Table 3 performs similar model specification experiments with reference to our main model 10, which is also pasted for readability of the table. We start by testing the effects of removing abnormal percentage volume from the model in what we dub as model 10-no volume. In fact, this may be taken as a version of Schaller and van Norden (2002), augmented to include the effects of monetary policy as measured by the FFR. In this case, the LRT applied to restricting model 10 gives a test statistic of 111.62 ((= 2 x (-2094.307 + 2150.115)) which, under a Chi-square distribution with 2 degrees of freedom (as the restrictions concern $\gamma_{s,v} = \gamma_{c,v} = 0$), implies a rejection of the null of no difference in fit with a p-value of 0.000. This establishes, that once it is removed from the conditional mean function, abnormal volume does play a role in our exercises.[28] Similarly to what we have done above when transition probabilities are constant, in Table 3 we also test the restriction $\beta_{s,0} = \beta_{c,0}$, which we reject with a p-value of 0.017, and of $\sigma_s = \sigma_c$, which we reject with a p-value of 0.000. These further LRTs indicate that model 10 needs to be specified to include structurally different means and volatilities of returns even when there are no bubbles, no volume anomalies and in the absence of an active monetary policy.

5 Robustness Checks

In this Section, we experiment with a number of variations and extensions of the baseline switching regression framework covered in Section 4.2. The general feeling is that all such experiment will need to find that our key implications that the short-term rate has nonlinear effects on the size and duration of a bubble survive a number of additional tests and further specifications. Thus, confirming that monetary policy—for instance, through FFR changes—may locally (i.e., for very low rates) cause the inception of bubbles but globally (i.e., provided rates are

[28] In fact, models 9 and 10 as reported in Table 2 offer a chance to also test this restriction formally by computing the LRT based on deriving model 10 as a restriction (written as $\gamma_{q,vabs} = \gamma_{q,vsqr} = 0$) of model 9. The resulting LRT statistic is 2.14 ((= 2 x (-2093.237 + 2094.307)) and implies a p-value of 0.343 that allows us not to reject the null hypothesis that there is no significant loss of fit from removing abnormal volume from the logistic function).

raised high enough) "prickle" them, causing their (partial, at least) collapse. In particular, Section 5.1 shows that expanding the Taylor polynomial expansion to include a cubic term in the logistic transition probability function would not alter the performance of the switching regression model in terms of either fit or viability of its economic interpretation. In Section 5.2, we replace the FFR with Wu and co-authors' notion of shadow rate to show that our main empirical insights remain intact. Section 5.3 shows that our definition of abnormal percentage volume does not play a first-order role as a driver of our main empirical findings. Likewise, Section 5.4 reports that using in our modelling exercises either an adjusted, orthogonalized version of BW sentiment index (as we have done in Section 4) or its un-adjusted raw equivalent, makes no difference to our main insights. Because our approach requires that bubbles be estimated as if these were easily measurable, in Section 5.5 we show that if the concept of bubble used in the rest of the paper were to be replaced by the methodology proposed by Campbell and Shiller (1987), our main results would go through intact and they would not be weakened. Finally, in Section 5.6, we check that our main results do not derive from our use of bubble estimates that occasionally turn negative.

5.1 A Cubic Term in the Logistic Probability Specification

Table 3 shows the results of the estimation of an extended switching model 10 (for simplicity, simply dubbed as 10') in which the logistic transition probability of a bubble surviving (i.e., not collapsing) and the short rate contain a cubic term:[29]

$$\begin{aligned} & q(|b_t|, b_t^2, |b_t|^3, |V_t^x|, (V_t^x)^2, |sr_t|, sr_t^2, |sr_t|^3) \\ &= \ell\big(\beta_{q,0} + \beta_{q,babs}|b_t| + \beta_{q,bsqr}b_t^2 + \beta_{q,bcube}|b_t|^3 \\ &+ \gamma_{q,vabs}|V_t^x| + \gamma_{q,vsqr}(V_t^x)^2 + \psi_{q,mabs}|sr_t| + \psi_{q,msqr}sr_t^2 + \psi_{q,mcube}|sr_t|^3\big), \qquad (12) \end{aligned}$$

Even though model 10' keeps performing better than the corresponding model 7' that excludes monetary policy altogether (e.g., the HQ criterion declines from 5.134 to 5.129 when the short-term rate is added), the majority of the coefficients in the logistic transition equation stop being significant in model 10' compared to model 10. In particular, in contrast to model 10, none of the three $\psi_{q,m}$ coefficients in $\ell(\cdot)$ are precisely estimated in 10', an indication that our data can identify the coefficients of a high-order Taylor expansion applied to the logistic function only up

[29] We have also experimented with the inclusion of cubic terms in the (regime-specific) conditional mean equations and in particular in extending 10 to include also cubic terms for the abnormal volume but obtained an inferior in-sample fit and therefore worse (much larger) values of the HQIC. To provide a benchmark, Table 3 also reports on the outcomes of extending model 7 (that does not include the short-term rate) to include cubic terms, which we dub model 7'.

to an order of two and not higher.[30] Moreover, Panel (a) of Figure 4 shows that a 12-month moving average of the bubble filtered probabilities derived from model 10' follow the same general dynamics as those inferred from model 10 (as already reported in Figure 3), but they are spikier and more volatile (hence, they give stronger bubble indications) between the late 1960s and the early 1990s, they are essentially identical between the early 1990s and the GFC, but provide less incisive (but still, above the historical average) indications of the persistence of a bubble regime between 2009 and 2023.

Panels (a) and (b) (representing the same 3-D plots but visualised from alternative perspectives) in Figure 5 emphasise that a model characterised by higher powers in the specification of $\ell(\cdot)$ would give rather murky economic insights: the probability of a bubble's onset or persistence become a rather complex and non-linear function of b_t and sr_t in which bubbles are made more likely by both very low rates (approximately below 1%) and by FFR levels in the range of 2-3%, while also bubbles with a rather large relative size (between 60 and 100%) are more likely to persist vs. either intermediate size (between 20 and 50 percent of the fundamental valuations) or massive (in excess of 110% in relative terms) bubbles. Even though it would remain the case that very large, extreme bubbles are likely to collapse and that a sufficiently high FFR (say, of 5 percent and higher) is likely to prick bubbles irrespective of their initial size, the complexity of the model 10' and its difficult interpretation lead us not to pursue higher-order Taylor expansions in the specification of $\ell(\cdot)$ in the core analysis of the paper.

5.2 Replacing the Federal Funds Rate with the Shadow Rate

Table 5 reports ML estimation results when the short-term rate capturing the US monetary policy is depicted by Wu's shadow rate. Because the shadow rate time series is available only for a 1990-2022 sample, as we have described in Section 3, we have pasted Wu's data with a measure of the shadow rate that would have prevailed over the two additional, disjoint sub-samples 1954-1989 and March 2022 – December 2023 on the basis of a simple linear projection of the shadow rate on the FFR based on 1990-2022 sample for which the two timeseries are simultaneously available. The general findings of Table 5 closely mimic the results reported with reference to Table 2. First, the models ruling out an impact by the short-term rate are obviously bound to provide empirical results that are identical to those in Table 2. For instance, they confirm that in spite of the richer parameterisations they imply, the regime switching regression models that take into account the alternating states induced by the periodically collapsing, but rational

[30] Interestingly, the coefficients in the conditional mean function of model 10' are hardly impacted (vs. those in model 10) by adding cubic terms to the logistic specification of transition probabilities.

bubbles achieve lower HQ information criteria vs. simple, single-state regressions. Moreover, when in model 5 the shadow rate is added, the HQIC declines to 5.135, starting from 5.360 achieved by model 2. Second, inserting the shadow rate in the logistic transition probability function, provides a further reduction in HQIC compared to when $\ell(\cdot)$ just depends on the relative bubble size and/or on the abnormal percentage volume, for instance to 5.125 in model 9 vs. 5.130 in model 7. Third, the estimates of model 10 in Table 5 confirm a few key findings obtained in Section 4.2: in the conditional mean function, higher shadow rates reduce aggregate stock valuation in both regimes, but more strongly in a bubble regime; the absolute value of the shadow rate increases the probability of a bubble persisting (i.e., $\psi_{q,mabs} > 0$ and precisely estimated) but the square of the shadow rate reduces such a probability (i.e., $\psi_{q,msqr} < 0$), which configures a globally stable switching regression system. The global stability emerges with reference to the relative size of the bubble too, i.e., in a neighbourhood of zero (no bubble), bubbles self-sustain and their formation supports their further survival, but a threshold exists beyond which bubbles of increasing relative size cause the probability of collapse to rise towards one. Finally, also in the case of the shadow rate, using the orthogonalized BW's sentiment indicator to replace abnormal volume or to supplement it (models 11 and 12), fails to improve the HQIC by much, also because abnormal volume variables in the logistic transition probability function do not turn up statistically significant.

For instance, Panel (b) of Figure 4 compares the 12-month moving average of the filtered probabilities of a bubble occurring/persisting and inferred from model 10 vs. model 5, the latter including the shadow rate only in the conditional mean function and not within the logistic probabilities. The graph is qualitatively indistinguishable from Panel (b) in Figure 3, where the same comparison was performed but with reference to FFR used as a proxy of the short-term rate. Panel (c) of Figure 4 performs, in fact, a direct comparison between model 10 estimated using the FFR as in Section 4.2, and the version of model 10 that employs the shadow rate (integrated with its linear projected values): obviously, the resulting bubble probabilities are indistinguishable throughout with only two exceptions. A TVPr switching regression model informed by the shadow rate infers a (moving average) probability of a bubble that between 2002 and 2003 reaches 60% thus exceeding the probabilities inferred when FFR data are used. On the opposite, between 2012 and 2015, the inferred bubble probabilities when the shadow rate is used are lower (sensibly so in 2014, with a difference that almost achieves 20%), which probably reflects the fact that already in 2014 the shadow rate starts climbing up from the very negative values achieved early in that year, thus signalling a less expansionary monetary policy.[31] Finally,

[31] The minor differences in the probability moving averages detected in correspondence to 2022, when the shadow rate spikes up and changes sign becoming positive, have a similar interpretation.

the Panel (d) of Figure 4 is virtually indistinguishable from Panel (d) of Figure 3, when monetary policy was measured by the FFR, apart when the US shadow rate did turn negative, over the sub-periods 2009-2015 and then 2020-2021.

Panels (c) and (d) of Figure 5 show once more 3-D plots of how the probability of the system being in a persistent bubble would change when either the relative size of any initial, starting bubble or especially the shadow rate changes.[32] These two panels are essentially identical to those displayed in Figure 2, apart from minor differences in the location of the shadow rate threshold beyond which the bubble probability increases, decreases, or reaches a peak. In fact, Panel (c) shows that for annualised values of sr_t exceeding approximately 3% (or below -3%), the probability to transition to a bubble are zero; the peak probability of a bubble is achieved for an annualised shadow rate of approximately 1% and in this case this probability is very close to 1, quite irrespective of the initial value taken by the relative size of the bubble provided the latter remains in the customary range [-80%, +80%] which appears to be very similar to the [100%, +100%] reported in Figure 2, Panel (c).

In conclusion, estimation results obtained by replacing the FFR with the Wu's shadow rate appear to be generally very similar, both numerically and in terms of their possible interpretation, to make a full-blown exercise of analysis and interpretation similar to Section 4.2 rather pointless at this stage. Moreover, we need to remind ourselves that the values of the shadow rate employed in this Section are partially estimated from an auxiliary regression in charge of yielding fitted values for the shadow rate for periods for which Wu and co-authors have refrained from its calculation because some data (concerning options on interest rate futures) are missing. Nonetheless, the analysis herewith performed allows to be comfortable with the idea that little or nothing of importance would change if we had adopted the shadow rate as our key measure of sr_t earlier.

5.3 An Alternative Definition of Abnormal Trading Volume

In Section 4 and Sections 5.1 and 5.2, we have worked with a notion of abnormal trading volume based on the measurement of the deviation of volumes from a trailing, 12-month moving average of past S&P index volumes. It is then natural to wonder, whether the selection of a 12-month

[32] To favor comparisons between Figures 2 and 5, the plots only encompass non-negative values of the shadow rate. Clearly, the shadow rate can turn negative and historically it has reached a minimum monthly value of -0.25%. However, because the shadow rate enters the logistic transition function in absolute value the resulting probabilities are perfectly symmetric around zero. Moreover, it remains the case that because the shadow rate can achieve negative values, it tends to span lower values than the FFR and to imply a lower sample mean.

average may affect our key findings in any detectable way. To this purpose, we adopt a shorter, 6-month trailing notion of moving average on which to base our estimation of abnormal volume. Because our prior of abnormal volumes and BW's sentiment indicator competing for a role in our empirical model had been dispelled by the results obtained from model 11 in Section 4.2, we apply this robustness check to model 11, obtaining a new model 11', whose ML parameter estimates are reported in Table 4, where, for pure comparison, also the estimates of model 11, that have already appeared in Table 2, are copied. Visibly, replacing the definition of abnormal percentage volume based on a 1-year norm with a faster evolving one has little or no effects on the estimates of our switching regression framework, in the sense that all coefficients are identically signed, most of them fall within ± one standard deviation of the point estimates that had been reported in Table 2 and more importantly all the considerations concerning the global stability of the dynamic model hold (i.e., that the estimates of $\beta_{q,babs}$ and $\psi_{q,mabs}$ are positive, large and statistically significant while $\beta_{q,bsqr}$ and $\psi_{q,msqr}$ are negative, large and also precisely estimated). Moreover, the effects of monetary policy on the size and chances of occurrence/persistence of bubbles are intact and also similar to models 10 and 10'. In particular, even though abnormal percentage volume and BW's orthogonalized sentiment may be expected to be positively correlated, both variables play a role in identifying the conditional mean function, at least with reference to the bubble regime. However, like in the case of model 11, in 11' the abnormal volume variable fails to play a role as a driver of the transition probabilities. We have also estimated a version of model 7 that includes the 6-month trailing moving average abnormal volume and obtained results essentially identical to those in Table 2.[33] All in all, these essentially identical empirical estimates confirm that our definition of abnormal percentage volume does not play a first-order role and that our results shall be robust to its specification.

5.4 Using a Raw Sentiment Indicator

As previously discussed, so far, we have worked with a notion of BW sentiment indicator that is orthogonalized vs. a number of key macroeconomic aggregates, also to prevent sentiment to pick rational drivers behind the US stock prices. Nonetheless, because such an adjustment may appear arbitrary and subjective, in this Section we briefly assess the performance of model 11 (to be called 11'' in what follows) when the original, unadjusted BW sentiment indicator is used to replace its (most commonly used) orthogonal version. Table 4 contains the ML estimation results for this case. The two rightmost columns of Table 4 show results for model 11'', when we revert to use the definition of abnormal percentage volume based on a 12-month moving average

[33] This result is available upon request to the Authors.

benchmark but deploying the unadjusted sentiment index. Also in this case, a comparison with models 11 and 11'' fails to reveal any remarkable differences. In fact, in the estimated logistic probability function now $\hat{\beta}_{q,babs}$ is smaller than in Section 4.2 and it fails to be statistically significant. The resulting HQIC is intermediate between those achieved by models 11 and 11' but the changes induced by the shift to the unadjusted notion of sentiment are so minor that our results may well be considered robust to the details of this choice.

5.5 An Alternative Measure of Bubbles

A careful Reader may wonder about the specific role played by the method that we follow to isolate the relative size of the bubble, b_t, which is of course a crucial variable in our research design but that remains rather elusive to precisely measure. To test the robustness of our result to this crucial aspect of our research design, we follow the seminal work by Campbell and Shiller (1987) and its applications in Schaller and van Norden (2002) to estimate one alternative measure of b_t and next use it in our empirical estimations.

Campbell and Shiller take steps from the evidence and considerable likelihood of variation over time in expected dividend growth and develop a framework to capture the evidence about future dividend growth contained in the information set available to market participants. Given the simple present value model of stock market prices

$$P_t = E_t\left[\sum_{j=0}^{\infty} \frac{1}{(1+sr)^j} D_{t+j}\right]. \quad (13)$$

Define the innovation in stock price as $\eta_t \equiv P_t - E_{t-1}[P_t]$. The present value model implies that the innovation can be expressed in terms of observable variables as:

$$\eta_t = P_t - (1+sr)[P_{t-1} - D_{t-1}]. \quad (14)$$

Because $P_{t-1}R_t = (P_t - P_{t-1} + D_{t-1}) = \eta_t + sr[P_{t-1} - D_{t-1}]$, this implies that $\eta_t = P_{t-1}[R_t - sr] + srD_{t-1}$, which is the excess total return from investing in an asset with price P_{t-1} and in which dividends are received at the beginning of the period and re-invested for a period at the risk-free short-term rate.

If the present value model were true, then some linear function of the current price and the dividend would be the optimal linear forecast of future dividend changes. Intuitively, this is because the current price reflects all available information, so innovations (i.e., excess returns) are unpredictable. Campbell and Shiller (1987) define the "spread" (S_t) as the difference between price and a multiple of current dividends:

$$S_t \equiv P_t - \frac{1+sr}{sr} D_t. \quad (15)$$

It is easy to show that S_t is the optimal linear forecast of S_t^*, where S_t^* is a weighted average of future dividend changes:

$$S_t^* = \frac{1+sr}{sr}\left[\sum_{j=1}^{\infty} \frac{1}{(1+sr)^j} \Delta D_{t+j}\right]. \tag{16}$$

A variety of studies, including Campbell and Shiller (1987), have found that excess returns are predictable using past information. They offer that such additional information contained in past dividend changes and stock market prices can be incorporated by estimating a VAR(p) representation for ΔD_t and S_t (where both variables have been demeaned). As always, any VAR(p) can be re-written in companion form as a VAR(1):

$$\boldsymbol{z}_t \equiv \begin{bmatrix} \Delta D_t \\ \Delta D_{t-1} \\ \vdots \\ \Delta D_{t-p+1} \\ \Delta S_t \\ \Delta S_{t-1} \\ \vdots \\ \Delta S_{t-p+1} \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & \dots & a_p & b_1 & \dots & b_p \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1 & \dots & c_p & d_1 & \dots & d_p \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} \Delta D_{t-1} \\ \Delta D_{t-2} \\ \vdots \\ \Delta D_{t-p} \\ \Delta S_{t-1} \\ \Delta S_{t-2} \\ \vdots \\ \Delta S_{t-p} \end{bmatrix} + \underbrace{\begin{bmatrix} u_{1,t} \\ 0 \\ \vdots \\ 0 \\ u_{2,t} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\boldsymbol{v}_t} = \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{v}_t. \tag{17}$$

In our context, the value of such a VAR representation is that it allows us to form an optimal forecast of future dividend changes,

$$S_{t+1}^f \equiv E_t[S_{t+1}^*] = \frac{1}{sr}\boldsymbol{e}_1'\boldsymbol{A}\left[\boldsymbol{I}_{2p} - \frac{1}{1+sr}\boldsymbol{A}\right]^{-1}\boldsymbol{z}_t, \tag{18}$$

where $\boldsymbol{e}_1'$ is a row vector that picks out the forecast for ΔD_t. Through an algebraic manipulation, we can transform the expression for the spread into an expression for the fundamental price P_t, where the latter now incorporates the optimal linear forecast of future dividend changes based on past prices and dividends, S_{t+1}^f. Let:

$$P_t = S_t^f + \frac{1+sr}{sr}D_t \tag{19}$$

Therefore, we can define a second measure of deviations from fundamentals, namely:

$$b_t \equiv \frac{P_t^a - P_t}{P_t^a} = \frac{P_t^a - S_t^f - \frac{1+sr}{sr}D_t}{P_t^a}. \tag{20}$$

Figure 1 shows (using the boldfaced, red dashed curve) the estimate of b_t derived from equation (20). The pertinent comments have been already expressed in Section 3, also because the estimated relative size of the bubble obtained from equation (9) and (20) turn out to be highly correlated.

Table 6 presents our key empirical findings in the same format as in Tables 2-5. However, to favour comparisons with the models covered earlier, their numbering is preserved in Table 6, but the models' numbers now are given a suffix 2nd, to indicate that these models are estimated taking b_t to be measured with the alternative methodology illustrated above. The two columns to the left show that under the second method, the simple regression models fail to provide any explanatory power for aggregate stock returns, in the sense that the FFR fails to explain stock returns as much as the relative bubble size does. In fact, adding the FFR to the regression 1-2nd, somewhat increases the HQIC. The comparison between model 5-2nd and models 1-2nd-2-2nd emphasises the importance of adopting a switching regression approach (when going from 1-2nd to 5-2nd, the HQIC declines from 5.43 to 5.24) and of including FFR in the conditional mean function (when going from 4-2nd to 5-2nd, the HQIC declines from 5.2383 to 5.2277), where it negatively correlates with S&P returns in at least one regime. Moreover, a comparison of models 5-2nd and 6-2nd shows that it is important to capture time variation in residual variances.

The key result comes nonetheless from the comparison between model 10-2nd where both b_t and sr_t enter the logistic transition probability function and model 7-2nd that excludes monetary policy from affecting the duration of the bubble regime. The decline in the HQIC is substantial (from 5.25 to 5.22) when going from model 7-2nd to model 10-2nd and the latter shows that FFR predicts lower S&P returns in both regimes, but in particular in the bubble regime. Even though the relative size of the bubble as part of the logistic function implies estimated coefficients carrying the same signs ($\beta_{q,babs} > 0$ and $\beta_{q,bsqr} < 0$) as in model 10 in Table 2, now such coefficients are imprecisely estimated. Yet, the effects of the FFR on the behaviour of the time-varying transition probabilities remains the one uncovered in model 10 and guarantees the global stability of the estimated model, as shown in Figure 6. In fact, Panel (a) in Figure 6 shows that regardless of the initial relative size of the bubble, for values of the FFR in excess of approximately 3% per year, bubbles have no chances to persist and they are expected to collapse. Moreover, Panel (b) emphasises once more that it is the FFR between 0.8 and 1.2 percent per year that maximise—in this case for a wide range of relative bubble levels between -200 and +200 percent, which is indeed unconditionally large in the light of the historical values of bubbles estimated using the second method—the probability of rational bubbles forming and persisting over time. Interestingly though, also in this case a Galí and Gambetti's (2014) effect kicks in: it is indeed visible that for values of FFR that decline below 0.6 percent and towards zero, the probability of bubbles declines, similarly to panels (c) and (d) of Figure 2.

Finally, Figure 7 shows 12-month moving averages of the filtered probabilities of a bubble under a variety of models when b_t is derived using the second method. Panel (a) shows that under this novel methodology not only the wedge between the bubble regime probabilities of models 10-

2nd and 5-2nd becomes massive and is due to the inclusion of monetary policy in the logistic function to estimate transition probabilities, but also that the probability of a bubble surges dramatically during the two QE episodes that have characterised recent history, i.e., 2009-2015, when the probabilities hovers around 0.75 and fails to decline much from the peak of 0.8 achieved during the sub-prime credit crisis of 2008, and 2020-2021, when the probability surges again to exceed 0.7 in the aftermath of the onset of QE policies to support markets in the midst of the pandemic shock. The role of monetary policy is well visible from the fact that under model 5-2nd the probability of a bubble stays rather low at a level between 0.20 and 0.25 which is only barely above the average probability of bubble conditions prevailing during the 1970s and 1980s. In fact, model 10-2nd is the implementation of model 10 that achieves the highest probability of a bubble prevailing over the entire 2009-2022 sample (as well as 2004-2005) as shown in Panel (b).[34] Panel (c) plots as usual the fitted (expected) S&P returns as well as the 12-month moving average of bubble regime probabilities, where the negative correlation between the FFR and the spikes in the probability of a bubble becomes rather obvious. All in all, Table 6 and Figures 6 and 7 provide support to our argument that our key insights are completely robust, in fact a portion of them might even come out strengthened, if we had adopted the approach by Campbell and Shiller (1987) to estimate the size and evolution of bubbles in the US stock market.

5.6 Dealing with Negative Bubbles

It is well known that rational bubbles cannot be negative: behaviourally, bubbles are rational in the sense that individuals do not mind paying a price over the fundamental price as long as the bubble element yields them the required rate of return the next period and is expected to persist. But if a bubble were negative, it would fall at a faster rate than the stock price. Hence a negative rational bubble ultimately ends in a zero price (say at time $t + N$). Rational agents realise this and they therefore know that the bubble will eventually burst. But by backward induction the bubble must burst immediately since no one will pay the 'bubble premium' in the earlier periods. Nonetheless, as shown in Figure 1, in Section 3 (as well as in 5.5) we have elected to use estimates of relative bubbles that can take negative values.

In this Section, we have instead estimated model 10 (when the FFR is included as a predictor of both the conditional mean and of the regime transition probabilities) under the first method of bubble estimation in equation (9) when b_t is restricted to be always non-negative. In essence, we

[34] Yet, model 10-2nd returns the lowest probabilities of a bubble prevailing between 1998 and 2001, which is sensible assuming that those years were characterised by the so called dot.com bubble but were well captured by the bubble measure inferred from the price-dividend ratio, which represents in essence our first method to estimate b_t.

simply adopt as a measure of relative bubbles $b_t^+ \equiv \min(0, b_t)$. We report model parameter estimates in the following (p-values are reported in the parentheses and boldfaced coefficients indicate statistical significance at 10 percent or less):

$$r_{t+1}^s = \underset{(0.804)}{-0.162} + \underset{(0.001)}{\mathbf{7.767}} b_t^+ - \underset{(0.000)}{\mathbf{0.120}} V_t^x - \underset{(0.001)}{\mathbf{4.931}} sr_t + \varepsilon_{s,t+1} \quad \sigma^s = \underset{(0.000)}{\mathbf{3.779}}$$

$$r_{t+1}^c = \underset{(0.000)}{\mathbf{1.487}} - \underset{(0.748)}{0.388} b_t^+ + \underset{(0.000)}{\mathbf{0.068}} V_t^x - \underset{(0.089)}{\mathbf{0.741}} sr_t + \varepsilon_{c,t+1} \quad \sigma^c = \underset{(0.000)}{\mathbf{2.371}}$$

$$q(|b_t|, b_t^2, |sr_t|, sr_t^2) = \ell\left(\underset{(0.459)}{-0.345} + \underset{(0.017)}{\mathbf{8.681}} |b_t^+| - \underset{(0.006)}{\mathbf{21.646}} (b_t^+)^2 + \underset{(0.001)}{\mathbf{4.855}} |sr_t| - \underset{(0.011)}{\mathbf{2.513}} sr_t^2\right)$$

Importantly, the signs and the general distribution of the levels of statistical significance across coefficients are identical to those obtained in Section 4.2 and reported in Table 2. For instance, in both models a large bubble increases expected stock returns in the surviving bubble regime (in fact, with a larger coefficient when b_t^+ replaces b_t) and it has a non-linear effect on the transition probability of a bubble regime, characterised by a significantly positive linear coefficient (8.68 here vs. 9.08 in Table 2) but a negative and also precisely estimated quadratic coefficient (-21.65 here vs. -19.37 in Table 2). Moreover, and crucially, under both b_t and b_t^+, we find that the short-term rate reduces expected stock returns in both regimes but yields a quadratic, concave parabolic effects on the bubble survival regime probabilities with significant coefficients of 4.86 and -2.51 (these were 4.50 and -2.22 in Table 2). In conclusion, these empirical findings show that none of our earlier results were driven by the fact that we had allowed bubbles to take a negative sign in spite of their presumed rational nature.

6 Out-Of-Sample Forecasting Performance: A Horse Race

The regime switching models entertained in our paper are relatively naive because they essentially consist of state-dependent regressions. Nonetheless, the dependence of probability of transition from a state of a persisting bubble to a state of collapse as a function of pre-determined factors gives them a strong non-linear character. It is well known that extremely non-linear models may occasionally over-fit the in-sample data and—also because a “misguided” positioning in the classical bias-variance trade-off space (i.e., by allowing excessive variance in exchange for no bias and a more precise in-sample fit)—offer a poor out-of-sample (OOS) performance. Therefore, in this Section we analyse the recursive OOS performance of a few of the most representative models treated in Section 4. Of course, we include in our analysis models 1, 5 and 10 that have marked the main logical steps of our analysis so far. Yet, it shall be instructive to include a few more models as well as a celebrated benchmark in the stock return predictability literature (see, e.g., Campbell and Thompson, 2008, and Welch and Goyal, 2008), the simple historical average (HA) model, by which $r^f_{t+1|t}(0) = t^{-1}\sum_{\tau=1}^{t} r_\tau$, where zero is the number assigned to the model HA.

In the following, Section 6.1 describes the recursive OOS experiment performed, Section 6.2 explains the economic and statistical performance measurements that we use to assess the predictive accuracy of our main models, results can be found in Section 6.3. Section 6.4 examines the empirical results of the economic value of our OOS exercise.

6.1 The OOS Recursive Experiment

We adopt a rather typical, recursive *pseudo*-OOS scheme. The "pseudo nature" of the experiment derives from the fact that we perform a horse race among the main models that have turned out to be able to provide a satisfactory trade-off between in-sample fit and parsimony (taken as an indicator of expected predictive performance) in Sections 4 and 5 using all the available data. Equivalently, because of obvious feasibility constraints, we refrain from doing afresh the model specification search along our selected OOS testing period to select the best models to be adopted and compared at each time iteration. Nonetheless, the experiment proceeds in a recursive fashion as an actual user of our regime switching predictive regressions would have done in real time. Starting with December 1999, at the end of every month we proceed to the estimation of versions of models 0, 1, 2, 5, 6, 7, 10 and 12 in which only values of the predictors (i.e., the relative size of the bubble, the FFR and the percentage abnormal volume) available as of that date are used to forecast the S&P returns of January 2000. The estimation is performed using all data available for the sample January 1955 – December 1999, for a total of 540 observations.[35] We then compute predicted returns ($r^f_{t+1|t}(M)$) for January 2000 according to model M and any economic decisions to be implemented between December 1999 and January 2000 (e.g., portfolio selection), the ensuing realised performance according to some economic loss function (e.g., realised returns as of the end of January 2000) and the one-month ahead forecast errors ($e^f_{t+1|t}(M) \equiv r_{t+1} - r^f_{t+1|t}(M)$). At this point, when a new vector of observations (also on the relative bubbles size, the FFR, and the percentage abnormal volume, besides $r_{2000:01}$) become available on January 2000, we perform afresh the estimation for the sample January 1955 – January 2000, for a total of 541 observations and compute predicted returns for February 2000 according to model M, economic decisions to be implemented between January and February 2000, the realised performance

[35] The December 1954 observation is lost because lagged variables are used both in the specification of the conditional mean and in the logistic probability functions. Note that with 540 observations, even the richly parameterised model 12 (with 17 parameters to be estimated) implies a saturation ratio of almost 32 which is not worrisome in terms of degrees of freedom.

according to an economic loss function and the one-month ahead forecast errors. We keep iterating on this recursive experiment until we exhaust the available data, which occurs in correspondence to November 2023, when we distil our last one-month ahead prediction.[36] This gives a total of 288 testable predictions, forecast errors and realisations of any adopted economic loss function. Because of its features, such a pseudo, recursive OOS experiment has an expanding window nature in the sense that while new observations are added, older observations are never dropped. This favours a progressive increase of the saturation ratio along the experiment to reach, even in the case of the most richly parameterised models (which is 12), a ratio in excess of 40, which ought to allow considerable precision in ML estimation.[37] Interestingly, our OOS period fails to overlap with Brooks and Katsaris' testing sample and also in that perspective gives a genuine out-of-sample assessment.

6.2 Economic and Statistical Performance Measures

Given that the recursive set up of the previous Section delivers T OOS prediction errors and realised values of some economic loss function, we briefly list the summary measures that we shall be using to compare alternative models and hence to set up an OOS horse race. As it is customary in this literature, our performance measures can be classified in two groups, those of purely statistical nature and those requiring an underlying logic of economic decision-making.

The statistical loss functions we experiment with are the classical squared and absolute value losses. The former leads to the root mean squared forecast error (RMSFE) criterion,

$$RMSFE(M,H) = \sqrt{T_H^{-1}\sum_{\tau=1}^{T_H}[e_{t+H|t}^{f}(M)]^2},$$

where $H \geq 1$is the forecast horizon of the exercise (in our case we focus only on the case of $H = 1$ month but there would be no problem with examining the comparative model performances at a range of horizons) and T_H is the number of OOS forecasts and associated errors derived from model M (in our application, $T_H = 288$ even though we can compute 289 forecasts because the

[36] Forecasts and the resulting economic decisions can also be computed with reference to December 2023, but at the time this paper was completed, our data set lacked information concerning January 2024, preventing the OOS assessment of the prediction error and of realised economic performance.
[37] Because our model is linear and foresees regimes and non-linear dynamics, adopting some rolling window scheme is hardly attractive because it would either reduce the saturation ratio well below the value of 40 achieved at the end of the experiment (for instance a rolling scheme based on a 10-year window would force us to single-digit saturation ratios) or ignore the adaptive learning of the nature of regimes that our models display, or both.

prediction for January 2024 cannot be assessed, as explained earlier). The absolute value loss leads instead to a mean absolute forecast error (MAFE) criterion:

$$MAFE(M,H) = T_H^{-1}\sum_{\tau=1}^{T_H} |e^f_{t+H|t}(M)|.$$

Another common measure of predictive accuracy popular in finance since Campbell and Thompson (2008) is the so-called OOS R-squared measure, which can be obtained as

$$R^2_{OOS}(M,1) = 1 - \frac{\widehat{Var}[e^f_{t+1|t}(M)]}{\widehat{Var}[e^f_{t+1|t}(HA)]},$$

where $\widehat{Var}[\cdot]$ is the sample variance over the recursive OOS period and the series of $e^f_{t+1|t}(HA)$ are defined as the one-month ahead forecast errors from the HA model, i.e., $e^f_{t+1|t}(HA) \equiv r_{t+1} - t^{-1}\sum_{\tau=1}^{t} r_\tau$. As it is well known in the predictability literature, a non-positive (positive) value for $R^2_{OOS}(M,H)$ indicates that $\widehat{Var}[(e^f_{t+1|t}(M))] \geq \widehat{Var}[(e^f_{t+1|t}(HA))]$ ($\widehat{Var}[(e^f_{t+1|t}(M))] < \widehat{Var}[(e^f_{t+1|t}(HA))]$), i.e., that a given model M cannot even (can) outperform the predictive power of a simple scheme that recursively estimates the sample mean of past stock returns. Yet, because $R^2_{OOS}(M,H)$ is based on a ratio of sample variances, this measure is intimately related to the use of squared loss functions.

Finally, we also summarise the forecasting accuracy of model M at horizon H through its mean percentage correct sign prediction statistic (the Success Rate, for short) computed as:

$$SR(M,H) = T_H^{-1}\sum_{\tau=1}^{T_H} I(r_{t+H}, r^f_{t+H|t}(M)) \qquad I\left(r_{t+H}, r^f_{t+H|t}(M)\right) = \begin{cases} 1 & if\ r_{t+H}\cdot r^f_{t+H|t}(M) > 0 \\ 0 & if\ r_{t+H}\cdot r^f_{t+H|t}(M) \leq 0 \end{cases}.$$

In other words, the indicator function picks up (taking a unit value) time t forecasts of time $t + H$ returns that carry the same sign as the actual, realised returns. Therefore $SR(M,H)$ is the percentage of time during the OOS testing period in which model M correctly forecasts the sign of returns. Of course, a minimal requirement is for any model M to express a $SR(M,H) > 0.5$, which means that a model ought to outperform a simple coin flipping device when it comes to predict the sign of returns.

We also compute and keep track of the realised performance obtained from the implementation of two alternative and yet simple portfolio strategies based on the idea of taking it seriously the S&P return forecasts computed under the same range of models and benchmarks alluded to earlier. The first strategy is a simple portfolio switching approach that follows Pesaran and

Timmermann (1995), in which the optimal weight to be allocated to the stock index at time t is determined as[38]

$$w_t(M) = \begin{cases} 1 & if\ r^f_{t+1|t}(M) > sr_{t+1|t} \\ 0 & if\ r^f_{t+1|t}(M) \leq sr_{t+1|t} \end{cases},$$

where $w_t(M)$ is the weight to be attributed to the S&P 500 index at time t under model M when the forecast is $r^f_{t+1|t}(M)$ and $r^f_{t+1|t}(M) > (\leq)\ sr_{t+1|t}$ indicates that the excess return is predicted to be positive (non-negative). Residually, $1 - w_t(M)$ is the weight allocated to the risk-free asset, which in this case is identified with the 1-month T-bill rate, also because this is an investible asset.[39] Note that $sr_{t+1|t}$ is the short-term riskless rate that applies between time t and t + 1 that is however already known at time t, when the portfolio weights are selected. The availability of a time series of S&P 500 return forecasts, $r^f_{t+1|t}(M)$ (t ranges between December 1999 and November 2023) naturally delivers a time series of optimal switching weights $w_t(M)$ and hence of realised portfolio returns, $w_t(M)r_{t+1} + [1 - w_t(M)]sr_{t+1|t}$ to span the interval January 2000 – December 2023. Because $r^f_{t+1|t}(M)$ may differ from the realised aggregate stock return, it is possible that ex-post $r_{t+1} \leq sr_{t+1|t}$ ($r_{t+1} > sr_{t+1|t}$) even though ex-ante the investor, under model M may have predicted $r^f_{t+1|t}(M) > sr_{t+1|t}$ ($r^f_{t+1|t}(M) \leq sr_{t+1|t}$) thus making one inappropriate selection of the asset class.

The second portfolio strategy implemented is of a classical, Markowitz's mean-variance style in which

$$w_t(M) = \frac{r^f_{t+1|t}(M) - sr_{t+1|t}}{\gamma \sigma^2_{t+1|t}}, \tag{21}$$

where $\sigma^2_{t+1|t}$ is the variance of the (excess) returns predicted for time t + 1 as of time t and γ is a coefficient of risk averse of the investor we model. Consistently with earlier literature (see, e.g., Campbell and Thompson, 2008; Rapach et al., 2010) and for additional realism given the nature of the asset allocation problem, we truncate $w_t(M)$ to be in [0, 1], i.e., we do not allow short-sales of stocks and levered equity positions. Moreover, $\sigma^2_{t+1|t}$ is simply set to equal the variance of the recursive residuals from model 0 computed over our sample as it expands from January 2000 until December 2023. Such simple, expanding window sample variance forecasts are applied

[38] The switching nature of the strategy should not be confused with regime switching characterisation of the collapsing bubble TVProb models estimated in Sections 4 and 5.
[39] In our sample the correlation between the FFR and the 1-month T-bill rate systematically exceeds 0.98 in all regimes and hence it would be made little or no difference to our main results if we had used the FFR instead of the government bill rates.

uniformly across all models for two reasons. First, because the benchmarks adopted are classical but also homoskedastic, we would like to compare the performance across models in their predictive power for the mean of S&P 500 returns only. Therefore, we decide to neutralise the predictive power of a portion of the models even though many of these are heteroskedastic models that may forecast time-varying variances.[40] Second, because it would be clearly possible to fine tune the conditional variance components of all models (benchmark included), by resorting to fairly complex predictive tools (such as regime switching GARCH, see, e.g., Marcucci, 2005) but this would change somewhat the goals and meaning of our recursive prediction exercise.

Also in this case, $1 - w_t(M)$ is the weight allocated to the risk-free asset. The availability of a time series of S&P 500 return forecasts, $r^f_{t+1|t}(M)$, naturally delivers a time series of optimal switching weights $w_t(M)$ and hence of realised portfolio returns, $w_t(M)r_{t+1} + [1 - w_t(M)]sr_{t+1|t}$ to span the interval January 2000 – December 2023. Even though a direct comparison is impossible because the optimal mean-variance weight formula in (21) corresponds to an interior solution of a constrained optimisation problem (see, for instance, Cuthbertson and Nitzsche, 2005), it is sensible to add that a switching strategy represents a case, stark case of mean-variance that can be obtained as $\gamma \to 0$ and an investor would only care for maximising expected portfolio returns. Equivalently, a switching strategy may describe optimal portfolio behaviour for near-risk neutral investors.

Both strategies are applied both with and without imposing transaction costs. In the former case, the costs include both fixed and variable components according to the formula:

$$I\{w_t(M) \neq w_{t-1}(M)\}0.025 + 0.15|w_t(M) - w_{t-1}(M)|$$

$$\text{where} \quad I\{w_t(M) \neq w_{t-1}(M)\} = \begin{cases} 1 & if\ w_t(M) \neq w_{t-1}(M) \\ 0 & if\ w_t(M) = w_{t-1}(M) \end{cases}$$

in which the fixed cost is 0.025 and applied in all periods in which there are trades affecting the portfolio weights irrespective of their size and the variable cost is 0.15 multiplied by the absolute value of the change in the portfolio weights. Both the structure and the entity of transaction costs are borrowed from earlier literature (see, e.g., Balduzzi and Lynch, 1999; Guidolin and Hyde, 2012) and appear to be rather conservative in the age of many zero-transaction costs being popularised. Clearly, transaction costs are zero in the absence of traders while their maximum is 0.175 when the portfolio switches abruptly from 0 to 100 percent invested in stocks or vice

[40] As a result, one may argue that the results that follow may be interpreted as a lower bound to the OOS predictive performance of the regime switching models with collapsing bubbles.

versa.[41] Even though it may be of some interest to also consider the case of ex-ante transaction costs (when trades that change portfolio weights are avoided in case incurring transaction costs may fail to lead to a maximisation of the mean-variance objective), we shall leave this extension to future research.

The realised OOS performances of these strategies (with and without application of ex-post transaction costs), are ranked and compared using a battery of standard performance measures: besides the mean realised strategy return and its standard deviation, we also report the corresponding Sharpe and Treynor ratios (the latter based on estimating a market model regression of the realised strategy returns on the excess returns on the S&P 500 index), their Jensen's alpha (estimated from the same market model regression) and the strategy turnover index:

$$Turnover(M) = T_H^{-1} \sum_{\tau=1}^{T_H} |w_{\tau+1}(M) - w_{\tau}(M)|.$$

The turnover index is useful to report in case a Reader may have in mind levels or a composition (in terms of fixed vs. variable) of the transaction costs different from the one we have applied earlier. Of course, among all performance indicators, the Sharpe ratio carries a prominent meaning in the case of the mean-variance strategy which (as it is well known), can be re-cast as a problem of Sharpe ratio maximisation.[42]

6.3 Empirical Results from Statistical Loss Functions

Table 7 uses the statistical measures of predictive accuracy introduced earlier to document the relative, realised OOS forecasting performance of a range of regime switching models. The measures of predictive accuracy are computed in relative terms: in the top panel, the RMSFE and MAFE of models 1, 2, 5, 6, 7, 10 and 12 are reported as a ratio of the statistics obtained for model

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[41] Given a standard, initial unit wealth, 0.175 may appear very large because we are studying the case of portfolio strategies concerning the S&P 500. However, empirically, in the case of the switching strategies, we observe that $|w_t(M) - w_{t-1}(M)| = 0$ for the majority of the periods considered, while in the case of the mean-variance allocations, $|w_t(M) - w_{t-1}(M)|$ tends to be non-zero but rather small for most periods and practically all models. For instance, in the case of the benchmark model 0, the average transaction cost paid is 0.1% in the case of the switching strategy and 2.55% in the case of mean-variance; in the case of model 10, the average transaction cost paid is 2.13% in the case of the switching strategy and 3.39% in the case of mean-variance. Because these transaction costs are plausible but also rather large, they imply that in our exercise, excessive and frequent trading is steeply penalised.

[42] Even though it is natural to reward models that—for each or both of the strategies implemented—deliver high mean returns, low volatility, positive large and significant Jensen's alphas and maximum Treynor ratio, the meaning of these performance measures over and above the Sharpe ratio is more ambiguous and these performance measures are reported here simply because they are standard in the applied portfolio management literature.

0; in the bottom panel, the RMSFE and MAFE of models 0, 2, 5, 6, 7, 10 and 12 are reported as a ratio of the statistics obtained for model 1. Model 0 is the historical average of returns, while model 1 consists of a single-state regression model. The last two rows are instead reported for all models in terms of absolute statistical measures. For each indicator of statistical realised OOS performance, we have boldfaced the model that achieves the highest accuracy.

The most striking finding in Table 7 is that all models, from the simplest, single-state regressions to complex TVProb switching models outperform model 0 both in terms of RMSFE, MAFE, success rate and R^2_{OOS}. Given the widespread evidence that even a simple arithmetic mean benchmark is hard to outperform in recursive OOS backtesting (see, e.g., Welch and Goyal, 2008), such a result is rather interesting. In particular, model 7 which features regime switching, collapsing bubbles but gives no role to monetary policy, at a ratio of 0.946, it minimises the RMSFE vs. model 0 and leads to a higher R^2_{OOS} (0.105 vs. zero, which holds by construction). Model 12, which also captures the impact of the FFR on the chances of bubbles to burst or persist, minimises at 0.927 the ratio of its MAFE vs. the one of model 0 and maximises the distance with the success ratio of model 0 (67.9% vs. 62.2%). These results imply that, depending on the specific loss function, the gain from using models that take the existence of collapsing bubbles and even the impact of monetary policy into account ranges from a 5 to an 8 percent reduction vs. model 0 and in an almost 6% increase in the fraction of correctly predicted signs of one-month ahead returns.

Table 7 also compares the performance of models 2, 5, 6, 7, 10 and 12 to model 1, which is a linear single-state regression which, nonetheless, takes bubbles and the FFR into account. Essentially, this portion of the table is about the OOS predictive power of models that incorporate regimes and the role of the FFR in modelling the TVProb. Also in this case, while model 7 outperforms under a square loss function by delivering a 2.5% reduction in RMSFE vs. model 1, model 12 does even better (with an improvement as large as 4.3%) under an absolute loss function. Moreover, the $R^2_{OOS} = 0.105$ for model 7 almost doubles the 0.058 scored by model 1, while the success rate of 67.9% of model 12 grossly exceeds the 64.6% characterising model 1.

One final comment concerns the fact that, even though the performance of model 10 which had become in many respects our reference model in Sections 4 and 5 as it did minimise the H-Q information criterion remains solid in Table 7, model 12 ends up yielding a superior predictive accuracy under the absolute value loss function that informs MAFE as well as a higher success ratio. Interestingly, model 12 is slightly worse than model 10 under a classical, squared loss function that informs a RMSFE-driven ranking. For instance, the ratio of MAFE of model 12 vs. the one of the historical mean is 0.927 vs. 0.94 for model 10, but the ratios of the RMSFEs are 0.957 which slightly exceeds 0.955, for models 12 and 10, respectively. Yet, because a portion of the H-Q criterion is based on a sum of squared residuals minimisation, the existence of such heterogeneity in the rankings of models under alternative loss functions appears to be unsurprising. All in all, we conclude that regime switching models that take into account the periodically collapsing nature of bubbles and—at least under some loss functions—the role played by monetary policy in affecting the probabilities of such a collapse provide not only a rather compelling fit to a long sample of S&P 500 return data but also display considerable predictive power when standard metrics are employed. This provides backing to the idea that our modelling efforts in Section 4 and 5 may have revealed important aspects of the data generating process.

6.4 Empirical Results on Economic Value

Table 8 shows performance results when no transaction costs are imputed. At a bird's eye view of the best performances, which are boldfaced for each type of strategy/imputed risk aversion (in the case of mean-variance)/portfolio indicator, what is probably expected from the previous Section is that it is the arithmetic average (model 0) and models 7 and 10 that emerge as superior vs. all other models. Model 0 minimises realised portfolio volatility and turnover but fails to achieve high Sharpe and Treynor ratios, while models 7 and 10 show the highest realised mean returns and Sharpe ratios, also in dependence of the assumed coefficient γ.[43] However, it is model 10, when also changes in the FFR play a role in predicting regime switches and the probability of any bubbles collapsing, that maximises both the OOS realised Sharpe ratios and the Jensen's alphas under mean-variance strategy implementations characterised by $\gamma = 2$ and 3 (reaching ratios in excess of 0.75 and significant alphas in excess of 0.40% per year). Model 7 outperforms instead for less risk averse strategies, i.e., when $\gamma \rightarrow 0$ and a simple switching portfolio scheme prevails and when $\gamma = 0.5$; of course, under these conditions of no or modest aversion to risk, model 7 achieves higher Sharpe ratios (in excess of 1 in annualised terms), even though the implied Jensen's alphas are in generally not precisely estimated.

What was less expected of Table 8 is that the relatively more complex model 12 would stop delivering any attractive realised OOS portfolio performance. Even though model 12 is never very distant from model 10, for instance in terms of realised portfolio means, volatilities and hence Sharpe ratios, it is evident that its additional parameters that made for a more complex non-linear dynamic behaviour end up detracting from its realised performance. Although finding modest

[43] Notably, the switching portfolio strategy based on the recursive sample mean return delivers a constant, full 100% commitment to the S&P 500 over time, so that the realised performance of model 0 is identical to that of the S&P index itself. The table also reveals that model 7 leads to the highest realised Treynor ratios for all strategies and assumed γ, as its estimated exposure to market portfolios risks are systematically lower than under model 10.

differences in the rankings of realised OOS quality in dependence of the specific choice of loss functions is unsurprising, in our application it is remarkable that this occurs when we switch from the statistical loss functions in Section 6.3 to the economic value ones implemented here.

Table 9 reports the empirical results on the same recursive portfolio results when transaction costs are imputed. Even though this occurs only on an ex-post basis, it is important to check on the effects of frictions because in Table 8 one can observe a massive difference between the realised OOS turnover of (say) models 7 and 10 vs. the benchmarks, models 0 and also 1. For instance, in the case of mean-variance strategies with $\gamma = 2$, model 10 yields a monthly average turnover of 2 percent vs. 0.1 percent in the case of model 0; in the case of the switching, mean-oriented strategy, model 7 is characterised by a turnover of 16 percent vs. 0.3 percent in the case of model 0. Even though turnover indices of 2-16 percent appear to be rather modest (also as a result of the very simple asset allocation problem the investor solves), one wonders how realised performance may suffer when transaction costs are considered. Nonetheless, even though most performance indicators decline as a result of imposing transaction costs on an ex-post basis, Table 9 reports results that are qualitatively identical to those in Table 8. On the one hand, almost by construction, model 0 results are essentially unaltered because the benchmark historical sample mean implies little trading needs. On the other hand, even though their scores obviously worsen, it remains the case that model 10 outperforms when applied to the choices of a relatively risk-averse investor with high γ, that model 7 instead prevails in the case of low or (limit) zero γ, and that model 12 is similar but worse vs. model 10, even though the distance between the two now widens somewhat. For instance, for $\gamma = 0.5$, we find annualised Sharpe ratios of 1.05, 1.00 and 0.34 for models 7, 10 and 0, respectively; these ratios are down vs. the values of 1.29, 1.17 and 0.53 found for the same models in Table 8. This means that transaction costs impose a Sharpe ratio sacrifice that ranges between 0.17 and 0.24 in annualised terms, which appears large but not unreasonable. The Sharpe ratio of model 12 drops from 1.16 to 0.96 but remains inferior to that reported for model 10.

The only arguable difference between Tables 8 and 9 is that in the latter, the transaction costs generally turn the estimated Jensen's alpha into negative values, especially for model 0 and in general for mean-variance portfolio returns when $\gamma = 3$. However, models 10 and 12 are the only ones for which alphas remain positive when the transaction costs are considered even under transaction costs. The finding that models 10 and 12 may be resilient and provide appreciable performances when investors are assumed to be relatively averse to risk is intuitively consistent with the idea that also in OOS tests, more complex models that consider bubble risk may be of increasing usefulness to more risk-averse decision makers. Intuitively, the latter will care more for the risk posed by bubbles suddenly collapsing.

Figures 8-11 provide visual documentation to the findings in Tables 8 and 9 and allow us to grasp the underlying optimal portfolio choices in a more practical way. All these plots concern the case in which transaction costs are considered but the corresponding figures before transaction costs are imputed appear to be qualitatively the same[44]. Table 8 shows the wildly different optimal weights across alternative models and especially according to the choice of the type of strategy, i.e., as a function of whether γ is positive or not.[45] In particular, it is evident how model 0 implies no variability of optimal weights and almost zero turnover in the case of the mean-variance strategy, and a constant 100% commitment to the S&P 500 index. Even assuming $\gamma \rightarrow 0$, it is also clear that model 7 implies some more portfolio switching vs. model 10 and this may justify why by being more active, the former portfolio overperforms the latter in Table 9. Visibly, the changes in mean-variance portfolio weights across models 7 and 10 are similar, but—probably because of the inclusions of the signals coming from the FFR for the bubble regime probabilities—the latter shows sharper variations that end up improving its performance, as shown in Table 9. For instance, after the burst of the (alleged) bubbles in early 2000, early 2009, and early 2020 (with the Covid pandemic crisis), under model 10 the weight allocated to the S&P index shoots up to 75, just less than 80 and 50 percent, respectively, while these upticks are 30, 45 and 35 percent only, respectively under model 7. The performance of the US stock market in mid-2000, mid-2009 and mid 2020 all validate that such a strong, bullish reaction would have brought considerable luck.[46]

Figures 9-11 need instead to be compared across different models. In general, when the different scales of the returns plotted are considered, the returns derived from the switching strategies appear to be considerably more volatile than those obtained under mean-variance, which is to be expected given that the former case corresponds to $\gamma \rightarrow 0$ and the investor will trade more aggressively and apply no risk-exposure reduction. Interestingly, the plots of realised portfolio returns in Figures 10 and 11 are similar, but these are less spiky in the latter case, when model 10 is used. Finally, as one would expect, in all these figures, the expanding window *monthly* Sharpe ratio computed on the portfolio realised performances is higher under the switching strategy vs. the mean-variance case, but the difference shrinks as we go from the clearly miss-

[44] All plots of results obtained assuming no transaction cost are available upon request to the Authors.
[45] To improve visibility, Figure 8 plots the case of $\gamma = 0.5$ even though Figure 9-11 present the case of $\gamma = 2$.
[46] Similarly, at the peak of the (alleged) bubbles in late 2000 and mid-2008, under model 10 the weight allocated to the S&P index is flattened to zero where it remains for a few months at least, while such declines are slower to occur and quantitatively more timid under model 7. The performance of the US stock market in 2001 and late 2008 also validate that a strong, bearish reaction would have generated non-negative excess returns (the excess returns are prevented from being positive because we do not allow short sales).

specified benchmark model 0 to model 7 and then 10, which is a sign of how easy it is to extract economic value from the latter irrespective of the specific assumptions one makes on the preferences of the investor.

7 Discussion and Conclusions

In this paper, we have used a wide array of monthly data sources to measure equity fundamentals, the stance of monetary policy and a number of alternative methods to measure the (relative, to fundamentals) size of bubbles to test how and whether the policies implemented by the Federal Reserve over a long (1954-2023) sample may have impacted aggregate rational, stock price bubbles. We find that this may have occurred in two ways: deflating their size by depressing expected returns when bubbles are inflating, according to a standard mechanism through which higher (risk-free) discount rates and the resulting weaker real economic outlook would slow down the rate of increase of stock prices; potentially impacting their probability of collapse (hence, their expected duration), according to the mechanism already illustrated in a rather different empirical set up by Galí and Gambetti (2014).

Our main empirical finding is that indeed monetary policy has strongly, historically affected the relative size and duration of equity bubbles in the US. We propose an estimate by standard maximum likelihood methods, Gaussian regime switching regression models in which transition probabilities are time-varying and affected by a variety of factors (likewise expected stock returns), including measures of the stance of the Fed monetary policy (measured by either the Federal Funds rate or by Wu and Xia's shadow rate) but also extended to include the initial, relative size of the bubble, abnormal percentage trading volume and a classical proxy for market sentiment due to Baker and Wurgler. The empirical model is obtained by log-linearization of standard asset pricing identities and picks up the mechanism of periodically collapsing, rational bubbles first introduced in the seminal paper by Schaller and van Norden (2002). Our empirical estimates show that, on the one hand, a higher short-term policy rate reduces expected stock returns and hence the size of any ongoing bubble irrespective of the regime in which the markets may start from on a given month. On the other hand, the effect of a higher short-term rate on the transition probabilities of the system is complex and non-linear because the data show a strong appetite for higher-order terms (powers) in the specification of the logistic transition probability function. In fact, locally, in correspondence to a zero-rate approximation point, the linear impact of the rate on the logistic probabilities is positive and large and this implies that a lower rate does reduce the chances of a bubble collapsing, thus increasing its duration, so that very low rates do indeed foster the formation and the growth of bubbles. Yet, as the short-terms rates grow higher, the quadratic term in the logistic function is estimated to be negative and large and this leads to bubbles to burst, which means that sufficiently high rates steer bubbles to collapse. Because a similar, nonlinear dynamics also characterises the impact of the initial, relative size of bubbles on their duration, the system is globally stable and features a sequence of bubbles forming and collapsing over time, everything in a completely rational way and in the absence of first-order arbitrage opportunities.

These results are robust to a range of robustness checks. In particular, expanding the Taylor polynomial expansion in the logistic transition probability function to include a cubic effect of short-term rates and relative bubbles does not alter the performance of the switching regression model in terms of either fit or viability of its economic interpretation. When we replace the FFR with the shadow rate, our main empirical insights remain intact. Our definition of abnormal percentage volume does not play a first-order role as a driver of our main empirical findings. Likewise, adopting in our exercises the un-adjusted version of BW sentiment index makes no difference to our main insights. Finally, when the measurement of bubbles is replaced by the methodology proposed by Campbell and Shiller (1987), our main results go through intact.

When the best fitting models that have emerged in Sections 4 and 5 are applied in a rather classical, recursive OOS exercise, we find that—irrespective of the loss function assumed and whether this had a statistical or economic nature—the regime switching models that consider the existence of periodically collapsing bubbles always outperform a number of benchmarks, such as the recursive sample mean of S&P 500 returns and simple and single-state regressions that feature bubble indicators. Yet, the specific loss function adopted turns out to be of more relevance to discriminate between models that account for the impact of monetary policy on regime switching, in the sense that the latter type of model performs the best only when absolute value loss or Sharpe ratio maximisation within a mean-variance framework are assumed.

Of course, many extensions of our framework can be envisioned. Among others, firstly, it would be interesting to pursue alternative measures of the relative size of bubbles stemming from either the application of different methodologies, from the use of alternative notions of fundamentals diverse from dividends (e.g., earnings, that are equally available in Bob Shiller's data), or both. Secondly, in this paper we have rather simplistically identified the stance of monetary policy with the levels and with the short-term rates that are directly controlled by the Federal Reserve or that have been estimated and re-constructed from the price of assets (e.g., interest rate futures options) that reflect market expectations of future actions of the central bank. However, a voluminous literature exists that has identified the changes in monetary policy with the unanticipated variations (shocks) in such rates. All these extensions and research avenues represent exciting directions of research and development. Thirdly, a genuine out-of-sample validation of the nonlinear dynamics uncovered in this paper would be very important, not only by using post-2023 data that have not been employed in the analysis so far, but also with reference to other central bank authorities and markets different from the US and the Federal Reserve system (e.g., the ones from the Eurozone and the UK).

References

Anderson, K., Brooks, C., & Tsolacos, S. (2011). Testing for periodically collapsing rational speculative bubbles in US REITs. *Journal of Real Estate Portfolio Management*, 17(3), 227-241.

Anderson, K., Brooks, C., & Katsaris, A. (2010). Speculative bubbles in the S&P 500: Was the tech bubble confined to the tech sector? *Journal of Empirical Finance*, 17(3), 345-361.

Baker, M., Wurgler, J., & Yuan, Y. (2012). Global, local, and contagious investor sentiment. *Journal of Financial Economics*, 104(2), 272-287.

Balduzzi, P., & Lynch, A. W. (1999). Transaction costs and predictability: Some utility cost calculations. *Journal of Financial Economics*, 52(1), 47-78.

Bernanke, B. S. (2010). Monetary policy and the housing bubble. Speech delivered at the Annual Meeting of the American Economic Association, Atlanta, Georgia. (http://www.federalreserve. gov/ newsevents/speech/bemanke20l00l03a.pdf).

Blanchard, O. J., & Watson, M. (1982). Bubbles, rational expectations and financial markets. Working Paper No. 945. National Bureau of Economic Research, Cambridge, MA.

Brooks, C., & Katsaris, A. (2005a). Trading rules from forecasting the collapse of speculative bubbles for the S&P 500 composite index. *Journal of Business*, 78(5), 2003-2036.

Brooks, C., & Katsaris, A. (2005b). A three-regime model of speculative behaviour: Modelling the evolution of the S&P 500 Composite Index. *Economic Journal*, 115(505), 767-797.

Brooks, C., Prokopczuk, M., & Wu, Y. (2015). Booms and busts in commodity markets: bubbles or fundamentals? *Journal of Futures Markets*, 35(10), 916-938.

Campbell, J. Y., & Shiller, R. J. (1987). Cointegration and tests of present value models. *Journal of Political Economy*, 95(5), 1062-1088.

Campbell, J. Y., & Thompson, S. B. (2008). Predicting excess stock returns out of sample: Can anything beat the historical average? *Review of Financial Studies*, 21(4), 1509-1531.

Cooper, I., & Maio, P. (2019). Asset growth, profitability, and investment opportunities. *Management Science*, 65(9), 3988-4010.

Cuthbertson, K., & Nitzsche, D. (2005). *Quantitative Financial Economics: Stocks, Bonds and Foreign Exchange*. John Wiley & Sons.

Cutler, D. M., Poterba, J. M., & Summers, L. H. (1991). Speculative dynamics. *Review of Economic Studies*, 58(3), 529-546.

Diba, B. T., & Grossman, H. I. (1988). The theory of rational bubbles in stock prices. *Economic Journal*, 98(392), 746-754.

Evans, G. W. (1991). Pitfalls in testing for explosive bubbles in asset prices. *American Economic Review*, 81(4), 922-930.

Galí, J., & Gambetti, L. (2015). The effects of monetary policy on stock market bubbles: Some evidence. *American Economic Journal: Macroeconomics*, 7(1), 233-257.

Goldfeld, S. M., & Quandt, R. G. (1976). *Studies in nonlinear estimation.* Cambridge: Ballinger.

Gordon, M. J. (1962). The savings investment and valuation of a corporation. *Review of Economics and Statistics*, 44(1), 37-51.

Greenspan, A. (2005). Economic flexibility. Remarks to the National Association for Business Economics Annual Meeting, Chicago, Illinois, September 27, 2005.

Guidolin, M., & Hyde, S. (2012). Can VAR models capture regime shifts in asset returns? A long-horizon strategic asset allocation perspective. *Journal of Banking and Finance*, 36(3), 695-716.

Hall, S. G., Psaradakis, Z., & Sola, M. (1999). Detecting periodically collapsing bubbles: a Markov-switching unit root test. *Journal of Applied Econometrics*, 14(2), 143-154.

Kindleberger, C. P. (1996). *Manias, Panics, and Crashes: A History of Financial Crises*. London: Macmillan.

LeRoy, S. F. (1989). Efficient capital markets and martingales. *Journal of Economic Literature*, 27(4), 1583-1621.

Liao, J., Peng, C., & Zhu, N. (2022). Extrapolative bubbles and trading volume. *Review of Financial Studies*, 35(4), 1682-1722.

Marcucci, J. (2005). Forecasting stock market volatility with regime-switching GARCH models. *Studies in Nonlinear Dynamics and Econometrics*, 9(4).

Nneji, O., Brooks, C., & Ward, C. (2013). Intrinsic and rational speculative bubbles in the US housing market: 1960-2011. *Journal of Real Estate Research*, 35(2), 121-152.

Pesaran, M. H., & Timmermann, A. (1995). Predictability of stock returns: Robustness and economic significance. *Journal of Finance*, 50(4), 1201-1228.

Psaradakis, Z., Sola, M., Spagnolo, F., & Spagnolo, N. (2009). Selecting nonlinear time series models using information criteria. *Journal of Time Series Analysis*, 30(4), 369-394.

Rapach, D. E., Strauss, J. K., & Zhou, G. (2010). Out-of-sample equity premium prediction: Combination forecasts and links to the real economy. *Review of Financial Studies*, 23(2), 821-862.

Roubini, N. (2006). Why central banks should burst bubbles. *International Finance*, 9(1), 87-107.

Schaller, H., & van Norden, S. (2002). Fads or bubbles? *Empirical Economics*, 27(2), 335-362.

Shiller, R. J. (2015). *Irrational Exuberance.* Revised and expanded third edition. Princeton University Press.

Taylor, J. B. (2013). *Getting Off Track: How Government Actions and Interventions Caused, Prolonged, and Worsened the Financial Crisis*. Hoover Press.

Van Norden, S., & Schaller, H. (1993). The predictability of stock market regime: evidence from the Toronto Stock Exchange. *The Review of Economics and Statistics*, 505-510.

Welch, I., & Goyal, A. (2008). A comprehensive look at the empirical performance of equity premium prediction. *Review of Financial Studies*, 21(4), 1455-1508.

Wu, J. C., & Xia, F. D. (2016). Measuring the macroeconomic impact of monetary policy at the zero lower bound. *Journal of Money, Credit and Banking*, 48(2-3), 253-291.

Table 1: Summary Statistics for Asset Returns and Predictor Variables of the Dataset

The table presents summary statistics for monthly returns on the S&P stock index, the associated abnormal trading volume, 1-month T-bills, the short-term (shadow) rate implied by interest rate options, the effective Fed funds rate, the price-dividend ratio for the S&P and the BW's sentiment index. The sample is June 1954 - December 2023. The shadow Federal funds (monetary policy) rate is extrapolated by a simple regression of the shadow rates estimated and made available by Cynthia Wu on the observed Fed fund rate with reference to a January 1990 – February 2022 sample (for which Wu's shadow rate is available). The sentiment index is Baker and Wurgler's (2006, 2007) measure available from Jeff Wurgler's personal web page.

	Index Returns	Abnormal Volume	1-m T-bill Rate	Shadow Rate	Shadow Rate (fitted)	Fed Fund Rate	Price/Div. Ratio	BW Sentiment	BW Sentiment (orthog.)
Mean	0.920	3.797	0.340	0.192	0.358	0.384	33.953	0.0003	-0.0002
Median	1.220	1.600	0.330	0.175	0.339	0.348	29.850	-0.100	-0.005
Maximum	12.320	112.300	1.350	0.678	1.529	1.592	51.050	3.040	3.210
Minimum	-20.190	-36.860	0.000	-0.249	-0.249	0.004	18.650	-2.360	-2.490
Std. Dev.	3.519	17.129	0.255	0.231	0.308	0.299	8.813	1.000	1.000
Skewness	-0.904	1.033	0.850	0.001	0.712	1.064	0.563	0.418	0.157
Kurtosis	6.626	5.883	4.049	1.890	4.127	4.581	1.700	3.587	3.873
Jarque-Bera	571.250	437.577	138.780	19.814	114.647	244.060	102.928	29.761	24.525
Probability	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Observations	835	835	835	386	834	834	835	684	684

Table 2: Main Results for Models in which Monetary Policy is Proxied by the Federal Funds Rate

The table presents the ML estimates of a range of regime switching models in which a selection of variables—including and excluding the FFR—are driving the probabilities of a regime shift from and to a bubble state. For most models, the sample is December 1954 - December 2023. When Baker and Wurgler's orthogonalized sentiment index is employed, the sample is instead July 1965 - June 2022. P-values are reported in the parentheses.

	Dependent variable: Returns																				
	Linear Single State			Switching Regressions						Time-Varying Probability Switching Regressions											
										without Fed Fund Rate				with Fed Fund Rate							
	(1)	(2)	(3)	(4)		(5)		(6)		(7)		(8)		(9)		(10)		(11)		(12)	
Regime				*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*
Intercept	**0.699**	**0.776**	**0.582**	**-1.146**	**1.227**	-0.640	**1.639**	-0.836	**1.579**	**-1.615**	**1.245**	-0.679	**1.508**	-0.082	**1.562**	0.023	**1.554**	0.107	**1.560**	0.082	**1.481**
	(0.000)	(0.001)	(0.035)	(0.013)	(0.000)	(0.325)	(0.000)	(0.278)	(0.000)	(0.008)	(0.000)	(0.206)	(0.000)	(0.895)	(0.000)	(0.967)	(0.000)	(0.802)	(0.000)	(0.838)	(0.000)
Bubble Size	**1.476**	**1.373**	**1.842**	**6.801**	-0.127	**6.474**	-0.646	**4.730**	0.045	**5.938**	-0.049	**2.607**	-0.061	**5.495**	-0.360	**5.432**	-0.362	**6.448**	-0.228	**7.031**	-0.142
	(0.004)	(0.017)	(0.005)	(0.000)	(0.806)	(0.000)	(0.250)	(0.001)	(0.937)	(0.000)	(0.925)	(0.045)	(0.941)	(0.000)	(0.571)	(0.000)	(0.578)	(0.000)	(0.746)	(0.000)	(0.853)
Abnormal % Volume	**0.014**	**0.014**	0.009	**-0.114**	**0.065**	**-0.117**	**0.066**	**-0.127**	**0.060**	**-0.107**	**0.061**			**-0.123**	**0.068**	**-0.122**	**0.068**	**-0.121**	**0.076**	**-0.117**	**0.078**
	(0.037)	(0.036)	(0.231)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)			(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Fed Fund Rate		-0.182	0.217			-1.491	**-0.971**	**-3.526**	**-0.844**					**-3.566**	**-0.949**	**-3.589**	**-0.918**	**-4.135**	**-1.016**	**-4.046**	-0.948
		(0.690)	(0.665)			(0.289)	(0.012)	(0.003)	(0.049)					(0.041)	(0.033)	(0.028)	(0.043)	(0.001)	(0.078)	(0.004)	(0.111)
Sentiment Index			**-0.358**									**-0.566**	-0.053					**-0.941**	0.059	**-1.071**	-0.018
			(0.019)									(0.099)	(0.773)					(0.008)	(0.695)	(0.006)	(0.906)
Volatility	**3.504**	**3.506**	**3.580**	**3.820**	**2.407**	**3.778**	**2.404**	**2.685**		**4.011**	**2.452**	**5.128**	**2.334**	**3.640**	**2.422**	**3.579**	**2.405**	**3.154**	**2.458**	**3.168**	**2.463**
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Probability Parameters:																					
Intercept				**-1.213**		**-1.258**		**1.632**		**0.928**		0.755		-0.265		-0.497		-1.511		**-1.407**	
				(0.000)		(0.000)		(0.000)		(0.091)		(0.346)		(0.713)		(0.341)		(0.115)		(0.031)	
abs(Bubble Size)										**11.357**		**15.273**		**9.230**		**9.083**		**7.393**		**7.872**	
										(0.004)		(0.030)		(0.016)		(0.016)		(0.083)		(0.052)	
abs(Abnormal % Volume)										-0.008				0.001				0.022			
										(0.756)				(0.975)				(0.564)			
abs(Fed Fund Rate)														**4.312**		**4.498**		**7.587**		**6.808**	
														(0.004)		(0.003)		(0.000)		(0.000)	
abs(Sentiment Index)												-1.055						0.503			
												(0.266)						(0.538)			
Bubble Size^2										**-22.581**		**-33.495**		**-19.582**		**-19.373**		**-16.256**		**-16.992**	
										(0.001)		(0.022)		(0.002)		(0.002)		(0.019)		(0.011)	
Abnormal % Volume^2										-1.410E-04				-1.780E-04				-4.840E-04			
										(0.639)				(0.612)				(0.374)			
Fed Fund Rate^2														**-2.182**		**-2.223**		**-3.821**		**-3.204**	
														(0.046)		(0.039)		(0.001)		(0.007)	
Sentiment Index^2												0.086						-0.398			
												(0.827)						(0.173)			
Observations	829	829	684	829		829		829		829		684		829		829		684		684	
Number of parameters	4	5	6	9		11		10		13		13		17		15		21		17	
Log likelihood	-2214.389	-2214.309	-1840.481	-2111.586		-2107.796		-2121.674		-2101.738		-1780.956		-2093.237		-2094.307		-1728.673		-1732.906	
Hannan-Quinn criterion	5.356	5.361	5.409	5.136		5.136		5.165		5.130		5.279		5.128		5.122		5.170		5.160	

Table 3: Restricted Models Used in Likelihood Ratio-Based Specification Tests

The table presents the ML estimates of a range of regime switching models in which a selection of variables—including and excluding the FFR—are controlling the probabilities of a regime shift from and to a bubble state. For all models the sample is December 1954 - December 2023. P-values are reported in the parentheses.

	Dependent variable: Returns															
	Switching Regressions								Time-Varying Probability Switching Regressions							
	(5)		(5 - one intercept)		(5 - fads model)		(5 - mixture model)		(10)		(10 - no volume)		(10 - one intercept)		(10 - homoskedastic)	
Regime	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*
Intercept	-0.640	**1.639**	**1.292**		**1.125**		-1.263	**1.342**	0.023	**1.554**	**-1.450**	**1.709**	**1.131**		0.127	**1.518**
	(0.325)	(0.000)	(0.000)		(0.000)		(0.108)	(0.000)	(0.967)	(0.000)	(0.057)	(0.000)	(0.000)		(0.771)	(0.000)
Bubble Size	**6.474**	-0.646	**4.648**	-0.159	**1.076**				**5.432**	-0.362	**2.121**	0.445	**3.816**	0.249	**3.985**	0.094
	(0.000)	(0.250)	(0.004)	(0.777)	(0.056)				(0.000)	(0.578)	(0.031)	(0.657)	(0.020)	(0.682)	(0.002)	(0.873)
Abnormal % Volume	**-0.117**	**0.066**	**-0.139**	**0.067**					**-0.122**	**0.068**			**-0.127**	**0.071**	**-0.131**	**0.066**
	(0.000)	(0.000)	(0.000)	(0.000)					(0.000)	(0.000)			(0.000)	(0.000)	(0.000)	(0.000)
Fed Fund Rate	-1.491	**-0.971**	**-4.580**	-0.510	-0.308				**-3.589**	**-0.918**	**2.526**	-0.811	**-5.249**	-0.285	**-5.543**	**-0.907**
	(0.289)	(0.012)	(0.000)	(0.202)	(0.460)				(0.028)	(0.043)	(0.043)	(0.105)	(0.001)	(0.509)	(0.000)	(0.059)
Volatility	**3.778**	**2.404**	**3.875**	**2.434**	**6.649**	**2.672**	**5.915**	**2.596**	**3.579**	**2.405**	**4.773**	**2.301**	**3.697**	**2.389**	**0.983**	
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	
Probability Parameters:																
Intercept	**-1.258**		**-1.343**		**-1.813**		**-1.604**		-0.497		-0.239		-0.670		**-0.919**	
	(0.000)		(0.000)		(0.000)		(0.000)		(0.341)		(0.717)		(0.227)		(0.075)	
abs(Bubble Size)									**9.083**		**33.078**		**8.831**		**11.435**	
									(0.016)		(0.000)		(0.024)		(0.004)	
abs(Abnormal % Volume)																
abs(Fed Fund Rate)									**4.498**		**-4.049**		**5.240**		**6.102**	
									(0.003)		(0.047)		(0.001)		(0.000)	
Bubble Size^2									**-19.373**		**-85.046**		**-19.141**		**-22.012**	
									(0.002)		(0.001)		(0.002)		(0.001)	
Abnormal % Volume^2																
Fed Fund Rate^2									**-2.223**		2.804		**-2.669**		**-3.272**	
									(0.039)		(0.121)		(0.015)		(0.002)	
Observations	829		829		829		829		829		829		829		829	
Number of parameters	11		10		6		5		15		13		14		14	
Log likelihood	-2107.796		-2112.089		-2173.030		-2169.532		-2094.307		-2150.115		-2097.131		-2104.036	
Hannan-Quinn criterion	5.136		5.141		5.270		5.257		5.122		5.247		5.124		5.140	

Table 4: Robustness Checks Concerning Cubic Terms, Alternative Measures of Volume and of Sentiment

The table presents the ML estimates of a range of regime switching models in which a selection of variables—including and excluding the FFR—drive the probabilities of a regime shift from and to a bubble state. For most models, the sample is December 1954 - December 2023. When Baker and Wurgler's orthogonalized sentiment index is employed, the sample is instead July 1965 - June 2022. P-values are reported in the parentheses.

	Dependent Variable: Returns													
	Time-Varying Probability Switching Regressions													
	without Fed Fund Rate				with Fed Fund Rate									
	(7)		(7') (cubic terms)		(10)		(10') (cubic terms)		(11)		(11') (6M volume)		(11'') (Raw BW Sentiment)	
Regime	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*
Intercept	**-1.615**	**1.245**	**-1.670**	**1.252**	0.023	**1.554**	0.032	**1.561**	0.107	**1.560**	-0.170	**1.817**	0.077	**1.537**
	(0.008)	(0.000)	(0.007)	(0.000)	(0.967)	(0.000)	(0.954)	(0.000)	(0.802)	(0.000)	(0.704)	(0.000)	(0.856)	(0.000)
Bubble Size	**5.938**	-0.049	**5.925**	-0.048	**5.432**	-0.362	**5.273**	-0.374	**6.448**	-0.228	**6.963**	-0.426	**6.605**	-0.174
	(0.000)	(0.925)	(0.000)	(0.927)	(0.000)	(0.578)	(0.001)	(0.572)	(0.000)	(0.746)	(0.000)	(0.560)	(0.000)	(0.804)
Abnormal % 12M Volume	**-0.107**	**0.061**	**-0.107**	**0.060**	**-0.122**	**0.068**	**-0.122**	**0.068**	**-0.121**	**0.076**			**-0.123**	**0.076**
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)			(0.000)	(0.000)
Abnormal % 6M Volume											**-0.138**	**0.080**		
											(0.000)	(0.000)		
Fed Fund Rate					**-3.589**	**-0.918**	**-3.653**	**-0.913**	**-4.135**	**-1.016**	**-3.735**	**-1.067**	**-3.871**	**-0.993**
					(0.028)	(0.043)	(0.027)	(0.071)	(0.001)	(0.078)	(0.004)	(0.055)	(0.003)	(0.087)
Sentiment Index (Orthogonal)									**-0.941**	0.059	**-0.901**	0.079		
									(0.008)	(0.695)	(0.015)	(0.592)		
Sentiment Index (Raw)													**-0.904**	0.039
													(0.014)	(0.795)
Volatility	**4.011**	**2.452**	**4.010**	**2.454**	**3.579**	**2.405**	**3.586**	**2.402**	**3.154**	**2.458**	**3.175**	**2.451**	**3.163**	**2.458**
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Probability Parameters:														
Intercept	**0.928**		0.615		-0.497		0.903		-1.511		-1.170		**-1.689**	
	(0.091)		(0.318)		(0.341)		(0.189)		(0.115)		(0.129)		(0.082)	
abs(Bubble Size)	**11.357**		**18.558**		**9.083**		**-16.422**		**7.393**		**6.946**		6.849	
	(0.004)		(0.022)		(0.016)		(0.048)		(0.083)		(0.098)		(0.107)	
abs(Abnormal % 12M Volume)	-0.008		-0.008						0.022				0.019	
	(0.756)		(0.757)						(0.564)				(0.633)	
abs(Abnormal % 6M Volume)											-0.002			
											(0.934)			
abs(Fed Fund Rate)					**4.498**		-5.098		**7.587**		**6.893**		**7.332**	
					(0.003)		(0.132)		(0.000)		(0.000)		(0.000)	
abs(Orthogonal Sentiment Index)									0.503		1.016			
									(0.538)		(0.238)			
abs(Raw BW Sentiment Index)													1.381	
													(0.110)	
Bubble Size^2	**-22.581**		**-54.367**		**-19.373**		51.516		**-16.256**		**-16.128**		**-15.572**	
	(0.001)		(0.0086)		(0.002)		(0.112)		(0.019)		(0.018)		(0.024)	
Abnormal % 12M Volume^2	-1.410E-04		-1.430E-04						-4.840E-04				0.000	
	(0.639)		(0.633)						(0.374)				(0.427)	
Abnormal % 6M Volume^2											0.000			
											(0.774)			
Fed Fund Rate^2					**-2.223**		3.442		**-3.821**		**-3.501**		**-3.686**	
					(0.039)		(0.564)		(0.001)		(0.003)		(0.002)	
Orthogonal Sentiment Index^2									-0.398		**-0.569**			
									(0.173)		(0.064)			
Raw BW Sentiment Index^2													**-0.725**	
													(0.025)	
[abs(Bubble Size)]^3			36.183				-36.435							
			(0.299)				(0.304)							
[abs(Fed Fund Rate)]^3							-0.604							
							(0.828)							
Observations	829		829		829		829		684		684		684	
Number of parameters	13		13		15		17		21		21		21	
Log likelihood	-2101.738		-2101.208		-2094.307		-2093.725		-1728.673		-1725.855		-1727.684	
Hannan-Quinn criterion	5.130		5.134		5.122		5.129		5.170		5.162		5.167	

Table 5: Robustness Checks Concerning Models in which Monetary Policy is Proxied by Wu's Shadow Rate

The table presents the ML estimates of a range of regime switching models in which a selection of variables—including and excluding the US shadow rate—are driving the probabilities of a regime shift from and to a bubble. For most models, the sample is December 1954 - December 2023. When Baker and Wurgler's orthogonalized sentiment index is employed, the sample is instead July 1965 - June 2022. P-values are reported in the parentheses.

	Dependent variable: Returns																				
	Linear Single State			Switching Regressions						Time-Varying Probability Switching Regressions											
										without Shadow Rate				with Shadow Rate							
	(1)	(2)	(3)	(4)		(5)		(6)		(7)		(8)		(9)		(10)		(11)		(12)	
Regime				*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*
Intercept	**0.699**	**0.792**	**0.638**	**-1.146**	**1.227**	-0.764	1.638	-1.111	**1.583**	**-1.615**	**1.245**	-0.679	**1.508**	-0.430	**1.598**	-0.296	**1.599**	-0.288	**1.664**	-0.254	**1.606**
	(0.000)	(0.000)	(0.012)	(0.013)	(0.000)	(0.232)	(0.000)	(0.114)	(0.000)	(0.008)	(0.000)	(0.206)	(0.000)	(0.478)	(0.000)	(0.572)	(0.000)	(0.531)	(0.000)	(0.533)	(0.000)
Bubble Size	**1.476**	**1.345**	**1.763**	**6.801**	-0.127	**6.541**	-0.675	**4.500**	0.054	**5.938**	-0.049	**2.607**	-0.061	**5.469**	-0.442	**5.379**	-0.456	**6.552**	-0.365	**6.983**	-0.353
	(0.004)	(0.019)	(0.007)	(0.000)	(0.806)	(0.000)	(0.250)	(0.001)	(0.922)	(0.000)	(0.925)	(0.045)	(0.941)	(0.001)	(0.479)	(0.000)	(0.476)	(0.000)	(0.588)	(0.000)	(0.629)
Abnormal % Volume	**0.014**	**0.014**	0.009	**-0.114**	**0.065**	**-0.116**	**0.066**	**-0.123**	**0.060**	**-0.107**	**0.061**			**-0.118**	**0.067**	**-0.119**	**0.068**	**-0.117**	**0.075**	**-0.114**	0.012
	(0.037)	(0.034)	(0.239)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)			(0.000)	(0.000)	(0.000)	(0.000)	(0.011)	(0.641)	(0.000)	(0.936)
Shadow Rate		-0.235	**0.102**			-1.259	**-1.028**	**-3.369**	**-0.884**					**-2.805**	**-1.044**	**-2.852**	**-1.020**	**-3.312**	**-1.199**	**-3.105**	**-1.148**
		(0.593)	(0.831)			(0.345)	(0.012)	(0.003)	(0.032)					(0.080)	(0.013)	(0.054)	(0.018)	(0.012)	(0.017)	(0.019)	(0.030)
Sentiment Index			**-0.347**									**-0.566**	-0.053					**-0.926**	0.069	**-1.027**	**0.077**
			(0.021)									(0.099)	(0.773)					(0.000)	(0.000)	(0.006)	(0.000)
Volatility	**3.504**	**3.506**	**3.581**	**3.820**	**2.407**	**3.787**	**2.398**	**2.675**		**4.011**	**2.452**	**5.128**	**2.334**	**3.763**	**2.408**	**3.689**	**2.388**	**3.338**	**2.425**	**3.322**	**2.421**
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Probability Parameters:																					
Intercept				**-1.213**		**-1.252**		**-1.655**		**0.928**		0.755		-0.375		-0.666		**-1.489**		**-1.540**	
				(0.000)		(0.000)		(0.000)		(0.091)		(0.346)		(0.595)		(0.217)		(0.099)		(0.019)	
abs(Bubble Size)										**11.357**		**15.273**		**9.625**		**9.560**		**8.134**		**8.514**	
										(0.004)		(0.030)		(0.012)		(0.012)		(0.054)		(0.033)	
abs(Abnormal % Volume)										-0.008				-0.003				0.013			
										(0.756)				(0.911)				(0.733)			
abs(Shadow Rate)														**5.059**		**5.209**		**8.164**		**7.264**	
														(0.003)		(0.002)		(0.000)		(0.000)	
abs(Sentiment Index)												-1.055						0.442			
												(0.266)						(0.588)			
Bubble Size^2										**-22.581**		**-33.495**		**-20.616**		**-20.506**		**-17.732**		**-18.323**	
										(0.001)		(0.022)		(0.001)		(0.001)		(0.010)		(0.005)	
Abnormal % Volume^2										-1.410E-04				-1.330E-04				-3.770E-04			
										(0.639)				(0.701)				(0.472)			
Shadow Rate^2														**-2.803**		**-2.814**		**-4.391**		**-3.650**	
														(0.026)		(0.022)		(0.001)		(0.005)	
Sentiment Index^2												0.086						-0.380			
												(0.827)						(0.198)			
Observations	829	829	684	829		829		829		829		684		829		829		684		684	
Number of parameters	4	5	6	9		11		10		13		13		17		15		21		17	
Log likelihood	-2214.389	-2214.245	-1840.552	-2111.586		-2107.349		-2121.395		-2101.738		-1780.956		-2092.077		-2093.166		-1727.764		-1731.987	
Hannan-Quinn criterion	5.356	5.360	5.409	5.136		5.135		5.164		5.130		5.279		5.125		5.119		5.167		5.158	

Table 6: Robustness Checks Concerning Models in which the Relative Bubble Size is Estimated Using Campbell and Shiller's Method

The table presents the ML estimates of a range of regime switching models in which a selection of variables—including and excluding the FFR—drive the probabilities of a regime shift from and to a bubble state. For all models the sample is February 1971 - December 2023. P-values are reported in the parentheses.

	Dependent variable: Returns											
	Linear Single State		Switching Regressions						Time-Varying Probability Switching Regressions			
									without Fed Fund Rate		with Fed Fund Rate	
	(1)	(2)	(4)		(5)		(6)		(7)		(10)	
Regime			*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*	*Bubble*	*Collapse*
Intercept	**0.908**	**1.004**	-0.675	**1.361**	-0.035	**1.794**	-0.095	**1.829**	-0.878	**1.467**	**0.707**	**1.702**
	(0.000)	(0.000)	(0.209)	(0.000)	(0.960)	(0.000)	(0.865)	(0.000)	(0.161)	(0.000)	(0.064)	(0.000)
Bubble Size	0.296	0.196	**2.373**	0.134	1.710	-0.252	2.070	-0.308	**2.402**	0.087	1.584	0.135
	(0.424)	(0.648)	(0.030)	(0.710)	(0.176)	(0.546)	(0.104)	(0.515)	(0.097)	(0.811)	(0.218)	(0.745)
Abnormal % Volume	0.001	0.001	**-0.120**	**0.064**	**-0.124**	**0.067**	**-0.145**	**0.073**	**-0.136**	**0.070**	**-0.129**	**0.076**
	(0.950)	(0.923)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Fed Fund Rate		-0.236			-1.908	**-1.014**	-0.322	**-1.540**			**-4.921**	**-1.061**
		(0.647)			(0.183)	(0.040)	(0.830)	(0.009)			(0.001)	(0.074)
Volatility	**3.638**	**3.640**	**3.954**	**2.511**	**3.856**	**2.495**	**2.877**		**4.110**	**2.476**	**3.181**	**2.579**
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)		(0.000)	(0.000)	(0.000)	(0.000)
Probability Parameters:												
Intercept			**-1.108**		**-1.135**		**-1.153**		**-1.308**		**-1.084**	
			(0.000)		(0.000)		(0.000)		(0.018)		(0.075)	
abs(Bubble Size)									-0.672		1.115	
									(0.717)		(0.599)	
abs(Abnormal % Volume)									0.015			
									(0.551)			
abs(Fed Fund Rate)											**7.020**	
											(0.000)	
Bubble Size^2									0.509		-0.971	
									(0.748)		(0.596)	
Abnormal % Volume^2									3.750E-05			
									(0.901)			
Fed Fund Rate^2											**-3.564**	
											(0.002)	
Observations	635	635	635		635		635		635		635	
Number of parameters	4	5	9		11		10		13		15	
Log likelihood	-1719.607	-1719.501	-1646.234		-1642.450		-1651.370		-1642.110		-1629.551	
Hannan-Quinn criterion	5.434	5.439	5.238		5.238		5.260		5.248		5.221	

Table 7: Statistical Performance Measures from a Recursive Out-of-Sample Forecasting Exercise

The table presents the relative, realised OOS forecasting performance of a range of regime switching models in which a selection of variables—including and excluding the FFR—drives the probabilities of a regime shift from and to a bubble state. For all models the sample is February 1971 - December 2023. The statistical measures of forecasting accuracy are reported in relative terms: in the top panel, the RMSFE and MAFE of models 1, 2, 5, 6, 7, 10, and 12 are reported as a ratio of the statistics obtained for model 0; in the bottom panel, the RMSFE and MAFE of models 0, 2, 5, 6, 7, 10, and 12 are reported as a ratio of the statistics obtained for model 1. Model 0 is the historical average of returns, while model 1 consists of a single-state regression model. The last two rows are instead reported for all models in terms of absolute statistical measures. For each indicator of statistical realised OOS performance, we have boldfaced the model that achieves the highest accuracy.

	Statistical Performance Measures							
						Time-Varying Probability Switching Regressions		
	Linear Single State			Switching Regressions		without Fed Fund Rate	with Fed Fund Rate	
vs. Model (0)	(0)	(1)	(2)	(5)	(6)	(7)	(10)	(12)
RMSFE	_	0.970	0.970	0.971	0.969	**0.946**	0.955	0.957
MAFE	_	0.969	0.953	0.954	0.954	0.947	0.940	**0.927**
vs. Model (1)								
RMSFE	1.030	_	0.999	1.000	0.999	**0.975**	0.984	0.986
MAFE	1.032	_	0.984	0.984	0.984	0.977	0.970	**0.957**
Success Rate	62.15%	64.58%	65.63%	65.63%	65.97%	67.36%	65.63%	**67.90%**
OOS R-squared	0.000	0.058	0.060	0.057	0.060	**0.105**	0.088	0.085

Table 8: Economic Value Performance Measures from a Recursive Out-of-Sample Forecasting Exercise, No Transaction Costs

The table presents the realised OOS portfolio performances of two strategies (switching and mean-variance) computed with reference to a range of regime switching models in which a selection of variables—including and excluding the FFR—drives the probabilities of a regime shift from and to a bubble state. For all models the sample is February 1971 - December 2023. The second column of the table also features results for the S&P 500 index, taken to represent a benchmark. The model providing the best performance is emphasised by boldfacing the corresponding result.

		Portfolio Performances without Transaction Costs																			
	S&P	Linear Single State												Switching Regressions							
		(0)				(1)				(2)				(5)				(6)			
Strategy		*switching*	*mean-variance*			*switching*	*mean-variance*			*switching*	*mean-variance*			*switching*	*mean-variance*			*switching*	*mean-variance*		
γ			*0.5*	*2*	*3*		*0.5*	*2*	*3*		*0.5*	*2*	*3*		*0.5*	*2*	*3*		*0.5*	*2*	*3*
Mean Return	7.68	7.68	2.50	1.84	1.77	9.69	3.44	2.08	1.93	10.13	3.99	2.22	2.02	10.09	3.90	2.20	2.01	10.30	3.78	2.17	1.99
Standard Deviation	13.03	13.03	**1.65**	**0.62**	**0.56**	10.18	1.87	0.65	0.57	11.54	2.33	0.72	0.60	11.64	2.33	0.72	0.60	11.61	2.30	0.72	0.60
α	_	**0.00**	0.01	0.002	0.002	0.36	0.10	0.03	0.02	0.31	0.13	0.03	0.02	0.30	0.12	0.03	0.02	0.32	0.11	0.03	0.02
		(0.00)	(0.003)	(0.001)	(0.001)	(0.11)	(0.02)	(0.01)	(0.004)	(0.09)	(0.03)	(0.01)	(0.005)	(0.09)	(0.03)	(0.01)	(0.004)	(0.09)	(0.03)	(0.01)	(0.004)
β	_	1.00	0.12	0.03	0.02	0.63	0.10	**0.02**	0.02	0.81	0.14	0.03	0.02	0.82	0.14	0.04	0.02	0.82	0.14	0.04	0.02
		(0.00)	(0.001)	(0.0002)	(0.0002)	(0.03)	(0.01)	(0.002)	(0.001)	(0.02)	(0.01)	(0.002)	(0.001)	(0.02)	(0.01)	(0.002)	(0.001)	(0.02)	(0.01)	(0.002)	(0.001)
Treynor Ratio (annual)	_	0.061	0.070	0.069	0.069	0.127	0.187	0.187	0.187	0.106	0.174	0.175	0.174	0.104	0.160	0.160	0.159	0.106	0.152	0.152	0.152
Sharpe Ratio (annual)	0.46	0.46	0.53	0.35	0.26	0.79	0.97	0.70	0.53	0.74	1.01	0.82	0.65	0.73	0.98	0.79	0.63	0.75	0.93	0.75	0.59
Turnover	_	**0.003**	**0.003**	**0.001**	**0.001**	0.27	0.09	0.02	0.02	0.19	0.10	0.02	0.02	0.16	0.09	0.02	0.02	0.16	0.09	0.02	0.01

		Time-Varying Probability Switching Regressions											
	S&P	without Fed Fund Rate				with Fed Fund Rate							
		(7)				(10)				(12)			
Strategy		*switching*	*mean-variance*			*switching*	*mean-variance*			*switching*	*mean-variance*		
γ			*0.5*	*2*	*3*		*0.5*	*2*	*3*		*0.5*	*2*	*3*
Mean Return	7.68	**11.90**	3.81	2.17	1.99	10.58	**4.36**	**2.31**	**2.08**	10.42	4.06	2.11	1.89
Standard Deviation	13.03	**9.65**	1.70	0.62	0.56	10.32	2.34	0.72	0.60	11.77	2.25	0.70	0.59
α	_	0.57	0.14	0.03	0.02	0.42	**0.16**	**0.04**	**0.03**	0.35	0.16	0.04	0.03
		(0.11)	(0.02)	(0.01)	(0.004)	(0.11)	(0.03)	(0.01)	(0.005)	(0.08)	(0.03)	(0.01)	(0.005)
β	_	**0.59**	**0.09**	0.02	0.02	0.65	0.13	0.03	**0.02**	0.84	0.13	0.03	0.02
		(0.03)	(0.01)	(0.002)	(0.001)	(0.03)	(0.01)	(0.002)	(0.001)	(0.02)	(0.01)	(0.002)	(0.001)
Treynor Ratio (annual)	_	**0.174**	**0.237**	**0.237**	**0.238**	0.137	0.204	0.204	0.204	0.106	0.198	0.198	0.199
Sharpe Ratio (annual)	0.46	**1.06**	**1.29**	0.88	0.65	0.87	1.17	**0.95**	**0.76**	0.76	1.16	0.93	0.74
Turnover	_	0.16	0.07	0.02	0.01	0.12	0.08	0.02	0.01	0.17	0.09	0.02	0.01

Table 9: Economic Value Performance Measures from a Recursive Out-of-Sample Forecasting Exercise, With Transaction Costs

The table presents the realised OOS portfolio performances of two strategies (switching and mean-variance) computed with reference to a range of regime switching models in which a selection of variables—including and excluding the FFR—drives the probabilities of a regime shift from and to a bubble state. The second column of the table also features results for the S&P 500 index, taken to represent a benchmark. The model providing the best performance is emphasised by boldfacing the corresponding result. The transaction costs are applied ex-post and they have the two-part (fixed and variable) structure described in the main text.

	Portfolio Performances with Transaction Costs																				
	S&P	Linear Single State												Switching Regressions							
		(0)				(1)				(2)				(5)				(6)			
Strategy		*switching*	*mean-variance*			*switching*	*mean-variance*			*switching*	*mean-variance*			*switching*	*mean-variance*			*switching*	*mean-variance*		
γ			*0.5*	*2*	*3*		*0.5*	*2*	*3*		*0.5*	*2*	*3*		*0.5*	*2*	*3*		*0.5*	*2*	*3*
Mean Return	7.68	7.68	2.19	1.54	1.47	9.13	3.00	1.76	1.63	9.73	3.53	1.90	1.71	9.75	3.45	1.87	1.70	9.96	3.34	1.84	1.68
Standard Deviation	13.03	13.03	**1.65**	**0.62**	0.56	10.20	1.86	0.65	0.57	11.54	2.33	0.72	0.61	11.64	2.32	0.73	0.61	11.61	2.30	0.72	0.61
α	_	-0.001	-0.02	-0.02	-0.02	0.31	0.07	-0.001	-0.01	0.27	0.09	0.01	-0.004	0.27	0.08	0.003	-0.01	0.29	0.07	0.001	-0.01
		(0.001)	(0.003)	(0.001)	(0.001)	(0.11)	(0.02)	(0.01)	(0.004)	(0.09)	(0.03)	(0.01)	(0.005)	(0.09)	(0.03)	(0.01)	(0.004)	(0.09)	(0.03)	(0.01)	(0.004)
β	_	1.00	0.12	0.03	0.02	0.63	0.10	**0.02**	0.02	0.80	0.13	0.03	0.02	0.82	0.14	0.03	0.02	0.82	0.14	0.03	0.02
		(0.0002)	(0.001)	(0.0004)	(0.0004)	(0.03)	(0.01)	(0.002)	(0.001)	(0.02)	(0.01)	(0.002)	(0.001)	(0.02)	(0.01)	(0.002)	(0.001)	(0.02)	(0.01)	(0.002)	(0.001)
Treynor Ratio (annual)	_	0.060	0.045	-0.028	-0.078	0.119	0.144	0.059	-0.001	0.101	0.143	0.082	0.040	0.099	0.130	0.072	0.032	0.102	0.122	0.063	0.023
Sharpe Ratio (annual)	0.46	0.46	0.34	-0.14	-0.28	0.74	0.74	0.21	-0.002	0.70	0.82	0.37	0.14	0.70	0.79	0.34	0.12	0.72	0.74	0.30	0.08
Turnover	_	**0.003**	**0.003**	**0.001**	**0.001**	0.27	0.09	0.02	0.02	0.19	0.10	0.02	0.02	0.16	0.09	0.02	0.02	0.16	0.09	0.02	0.01

		Time-Varying Probability Switching Regressions											
	S&P	without Fed Fund Rate				with Fed Fund Rate							
		(7)				(10)				(12)			
Strategy		*switching*	*mean-variance*			*switching*	*mean-variance*			*switching*	*mean-variance*		
γ			*0.5*	*2*	*3*		*0.5*	*2*	*3*		*0.5*	*2*	*3*
Mean Return	7.68	**11.57**	3.41	1.86	1.69	10.32	**3.95**	**2.01**	**1.79**	10.07	3.62	1.79	1.59
Standard Deviation	13.03	**9.64**	1.69	0.62	**0.56**	10.32	2.33	0.72	0.61	11.77	2.24	0.70	0.59
α	_	0.54	0.10	0.01	-0.002	0.40	**0.13**	**0.02**	**0.003**	**0.32**	0.12	0.01	0.0004
		(0.11)	(0.02)	(0.01)	(0.004)	(0.10)	(0.03)	(0.01)	(0.005)	(0.08)	(0.03)	(0.01)	(0.005)
β	_	**0.59**	**0.09**	0.02	**0.01**	0.65	0.13	0.03	0.02	0.84	0.13	0.03	0.02
		(0.03)	(0.01)	(0.001)	(0.001)	(0.03)	(0.01)	(0.002)	(0.001)	(0.02)	(0.01)	(0.002)	(0.001)
Treynor Ratio (annual)	_	**0.168**	**0.196**	0.107	0.044	0.133	0.175	**0.117**	**0.076**	0.102	0.166	0.104	0.060
Sharpe Ratio (annual)	0.46	**1.03**	**1.05**	0.38	0.11	0.84	1.00	**0.52**	**0.27**	0.73	0.96	0.47	0.21
Turnover	_	0.16	0.07	0.02	0.01	0.12	0.08	0.02	0.01	0.17	0.09	0.02	0.01

Figure 1: Behaviour of the Relative Size of Bubbles in the S&P Index Computed According to Two Alternative Methods

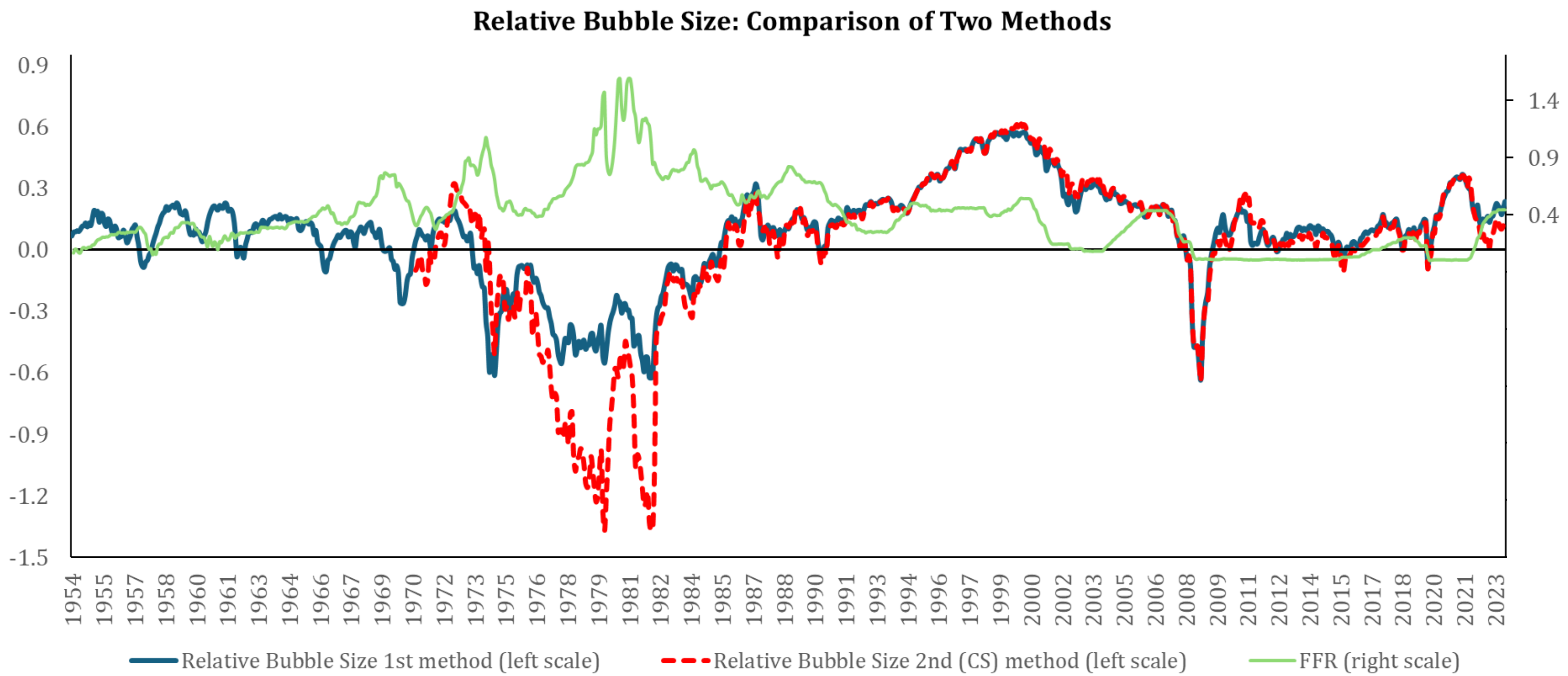


Figure 2: Behaviour of the Probability of a Bubble Surviving as a Function of the Relative Size of a Bubble and of the Short-Term Rate

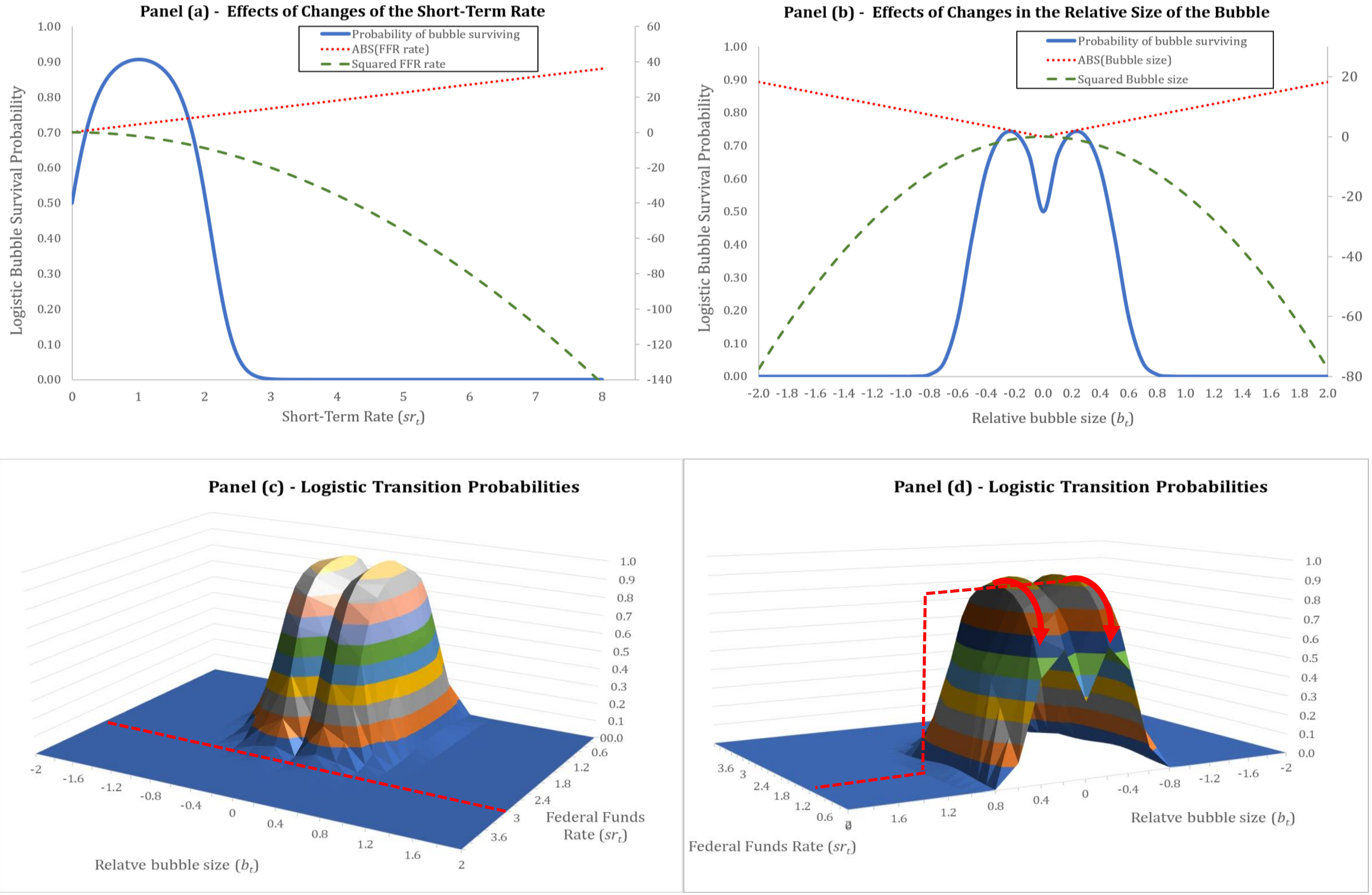


Figure 3: Comparing 1-Year Moving Averages of Real-Time Filtered Probabilities of a Bubble Regime Across Alternative Models

The shaded periods represent spells of quantitative easing policies. The solid, vertical bars denote end of alleged bubbles.

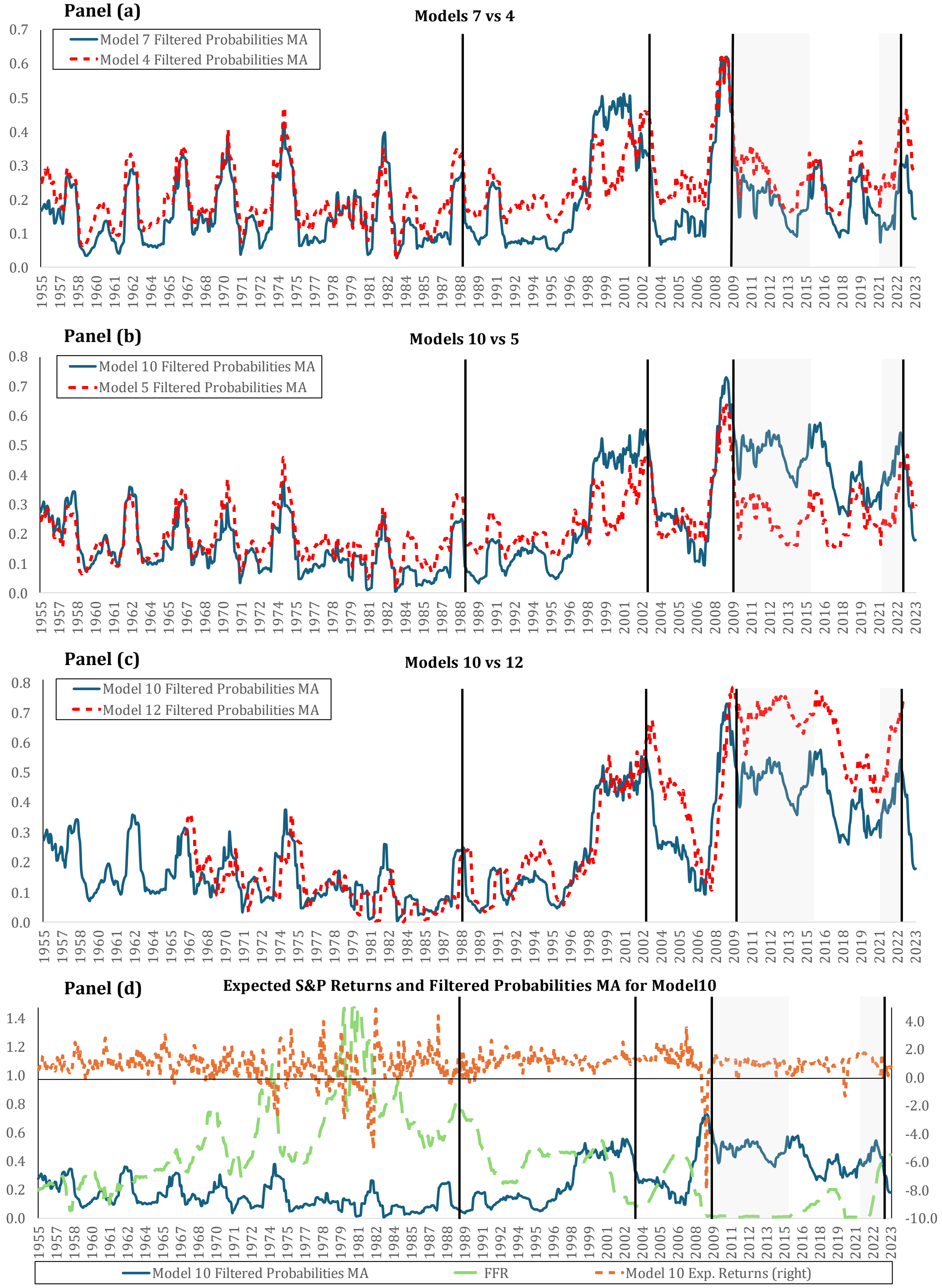


Figure 4: Comparing 1-Year Moving Averages of Real-Time Filtered Probabilities of a Bubble Regime Across Alternative Models: Robustness checks

The shaded periods represent spells of quantitative easing policies. The solid, vertical bars denote end of alleged bubbles.

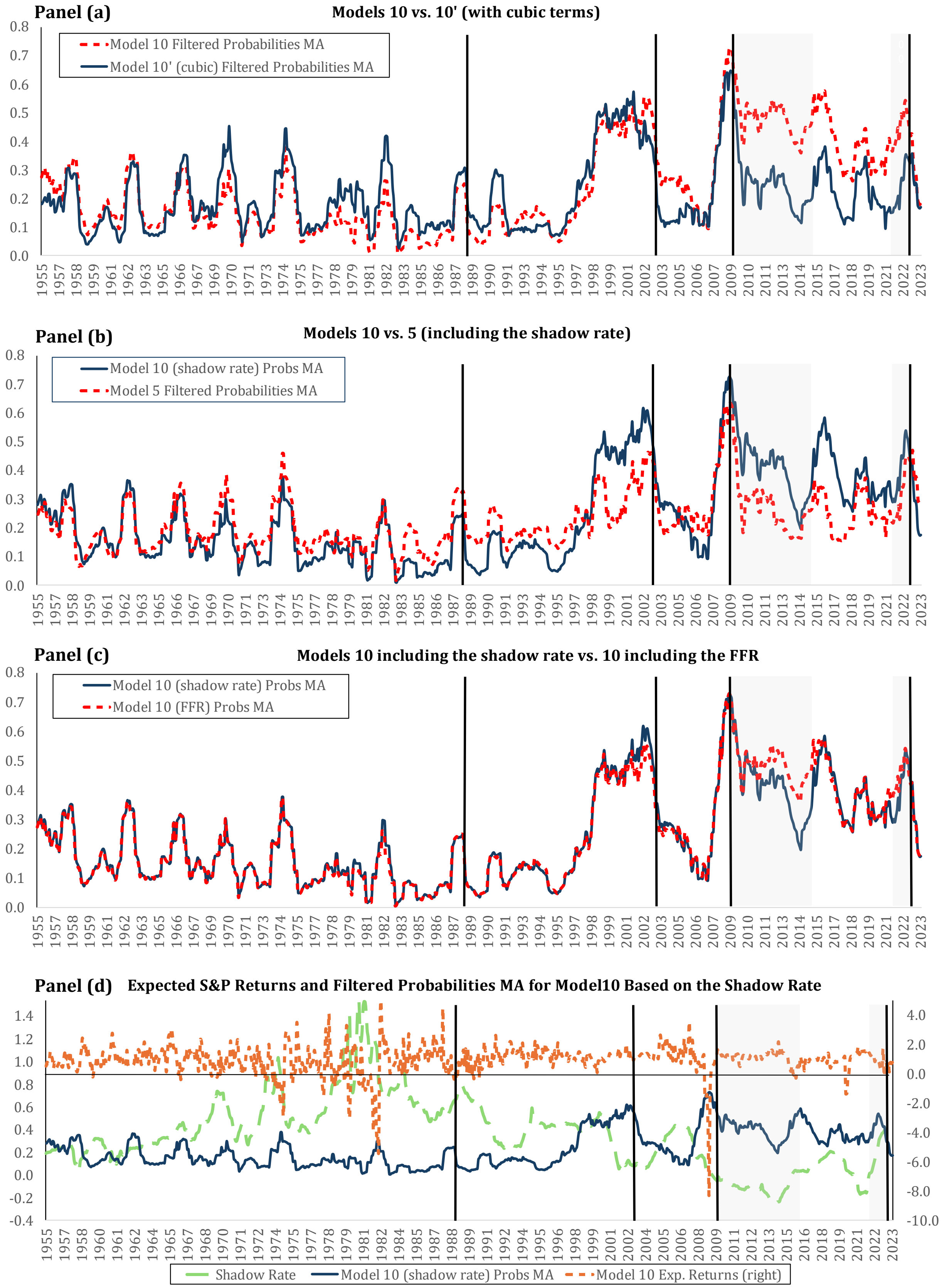


Figure 5: Behaviour of the Probability of a Bubble Surviving as a Function of b_t and of sr_t in Alternative Models

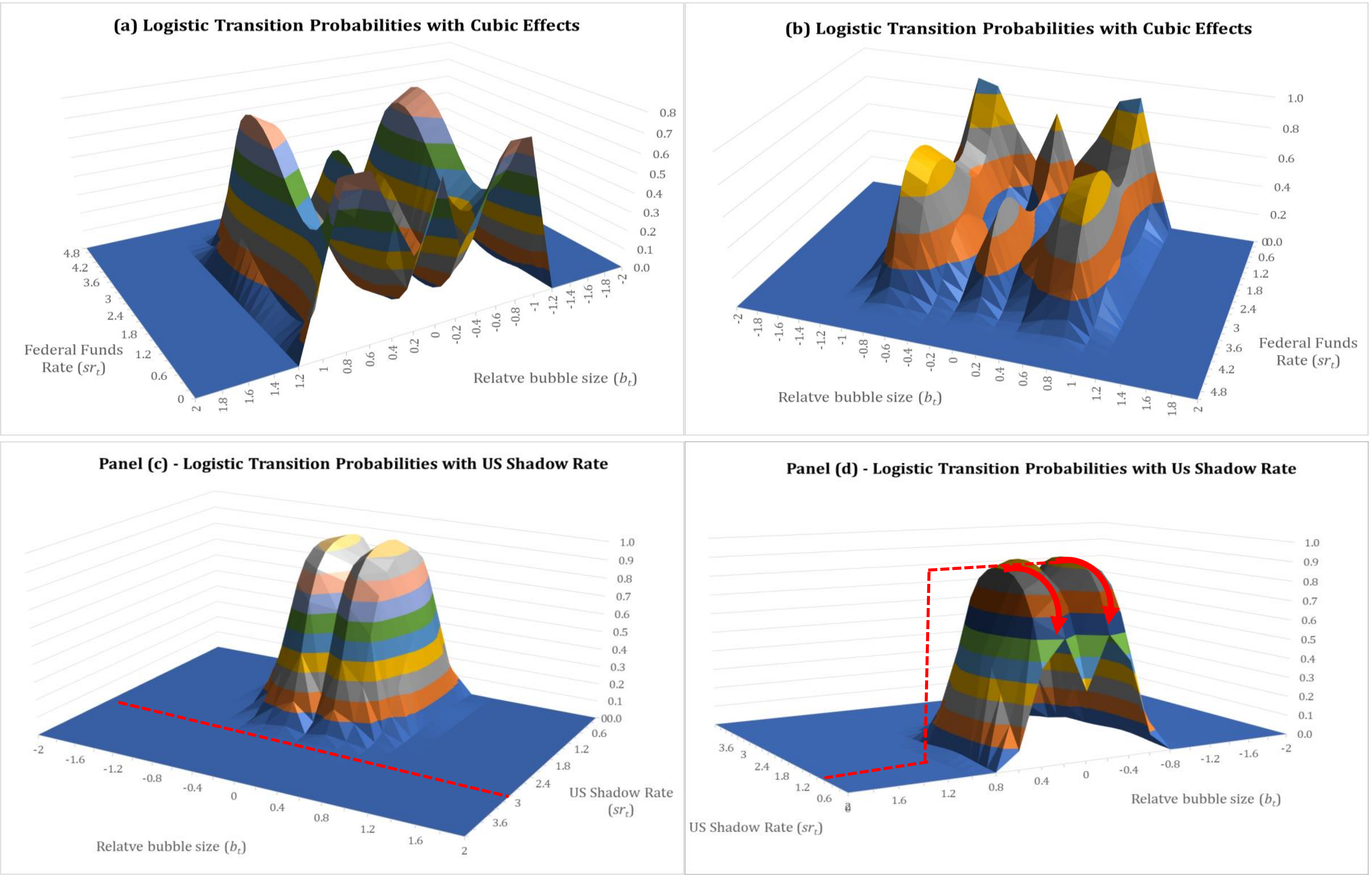
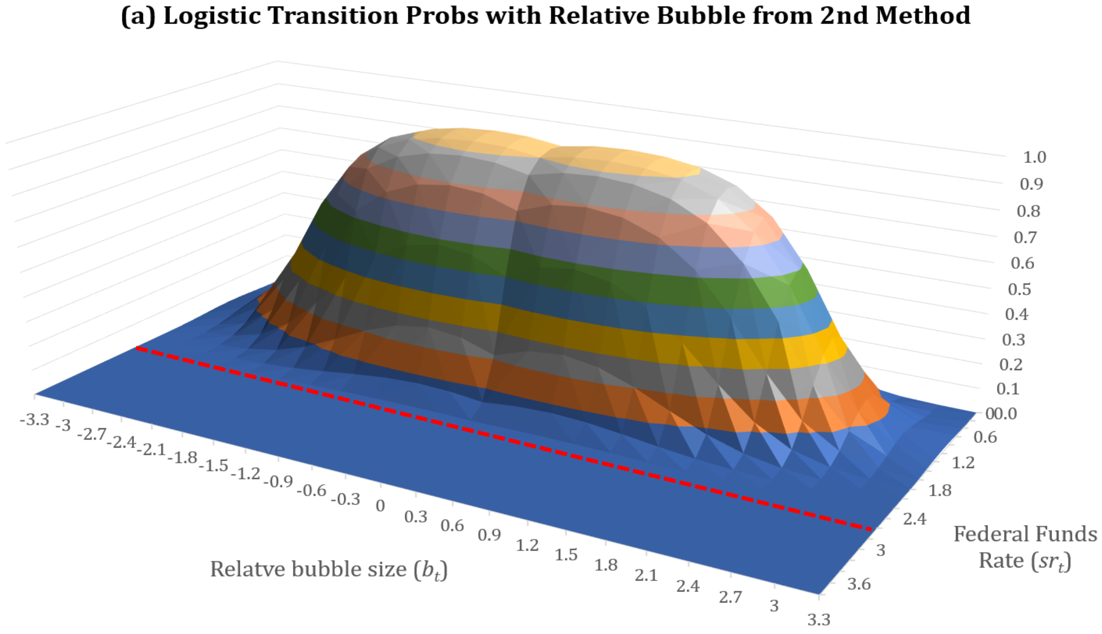


Figure 6: Behaviour of the Probability of a Bubble Surviving as a Function of b_t and of sr_t when the Relative Bubble is Estimated under the Second Method



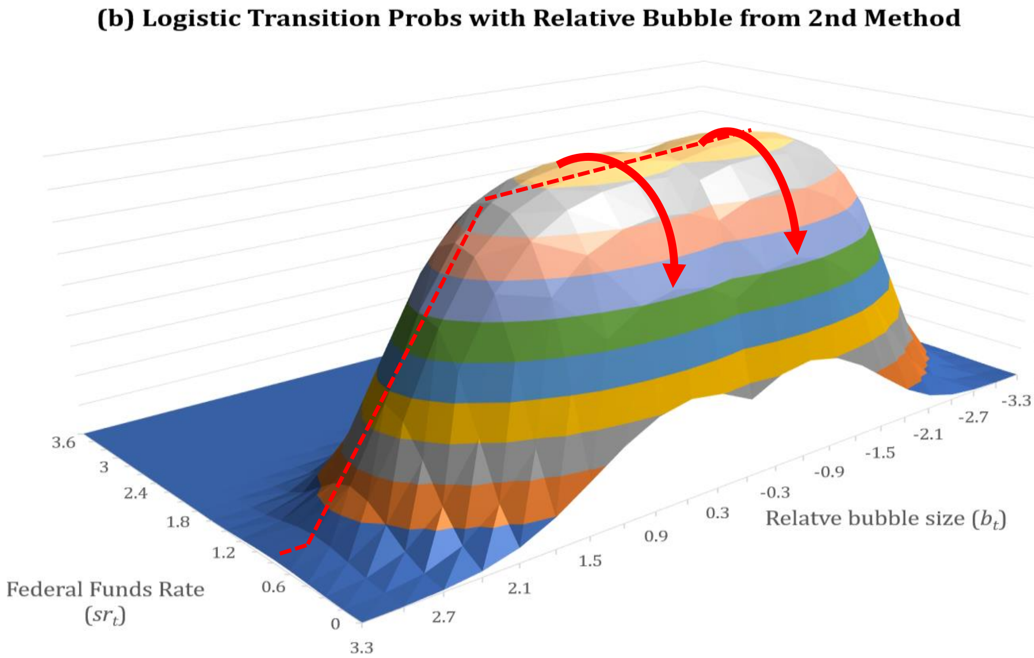


Figure 7: Comparing 1-Year Moving Averages of Real-Time Filtered Probabilities of a Bubble Regime Across Alternative Methods of Relative Bubble Estimation

The shaded periods represent spells of quantitative easing policies. The solid, vertical bars denote end of alleged bubbles.

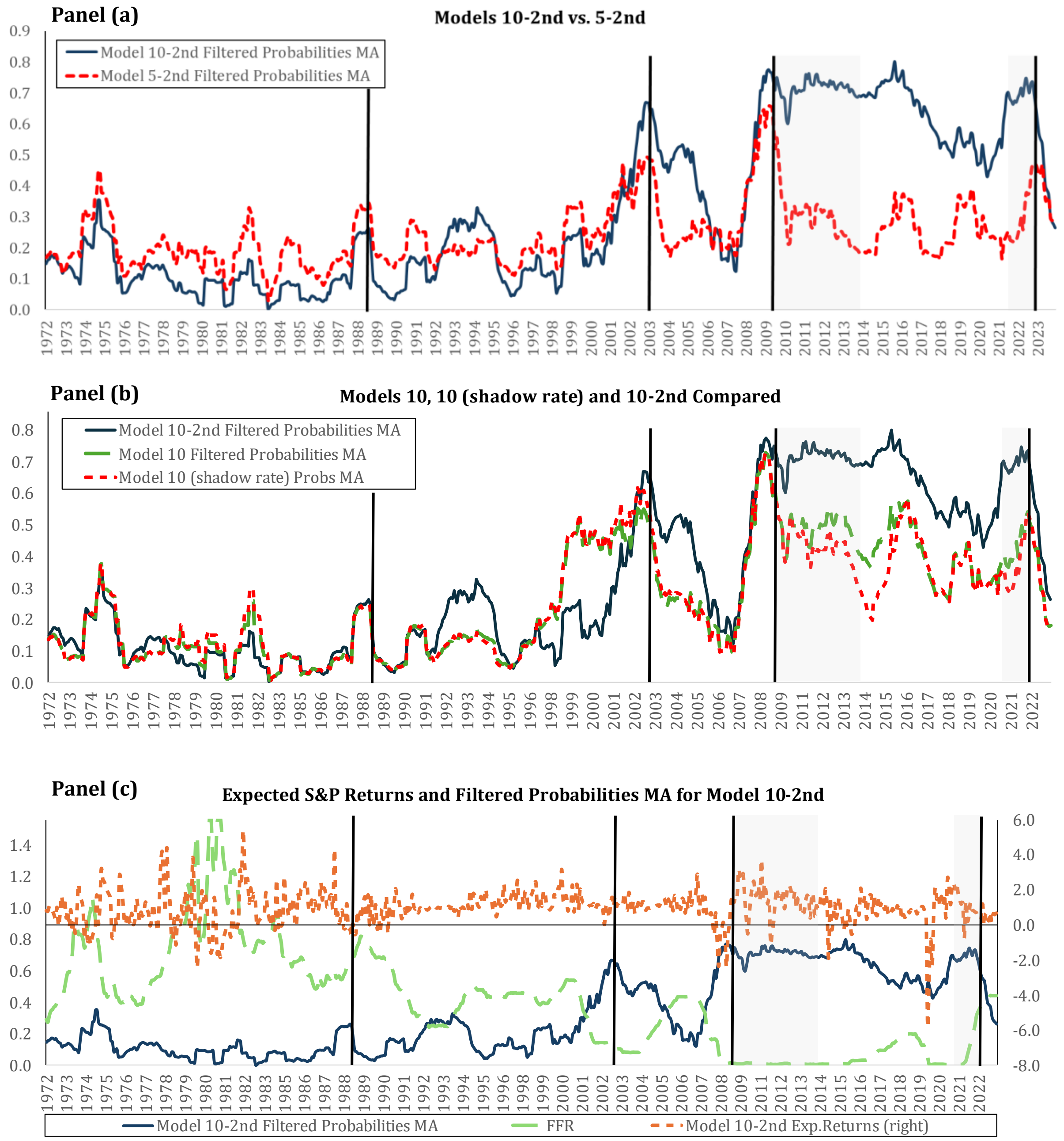


Figure 8: Optimal Portfolio Weights to the S&P 500 under Alternative Models for Switching and Mean-Variance (with $\gamma = 0.5$) Strategies, after Transaction Costs

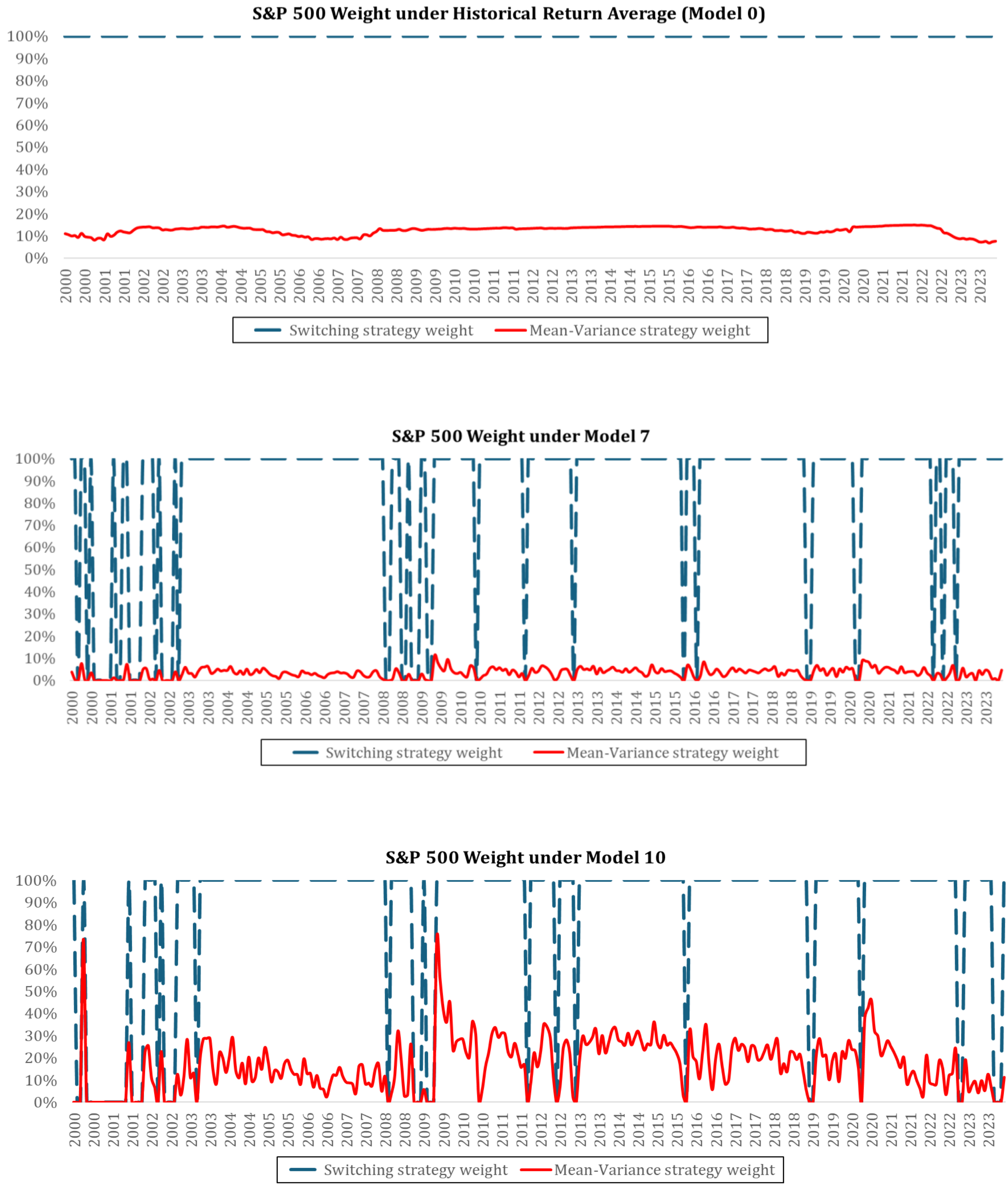


Figure 9: Portfolio Strategy Results under Benchmark Model 0 (Arithmetic Sample Mean) for Switching and Mean-Variance Portfolio (with $\gamma = 2$) Strategies, after Transaction Costs

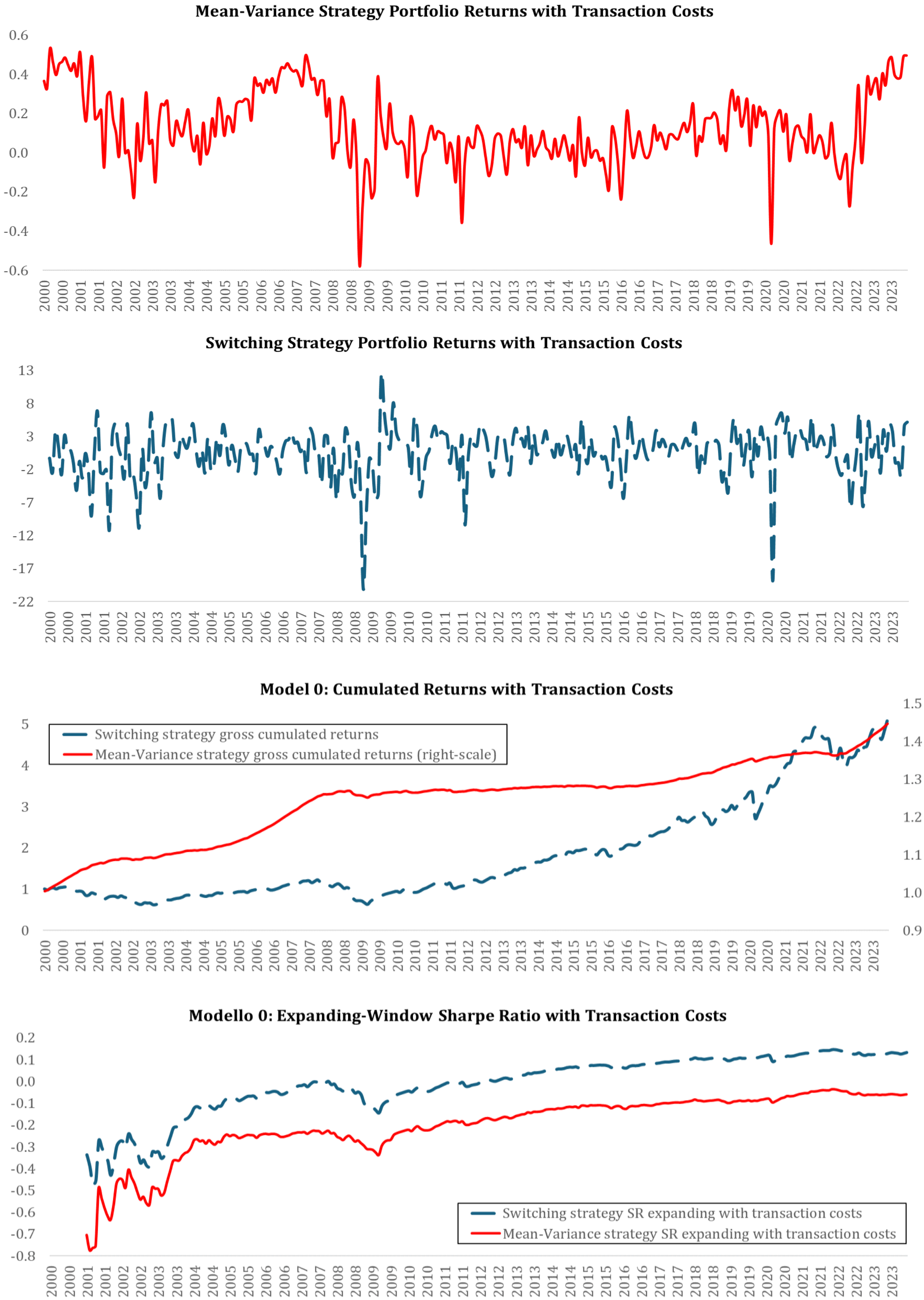
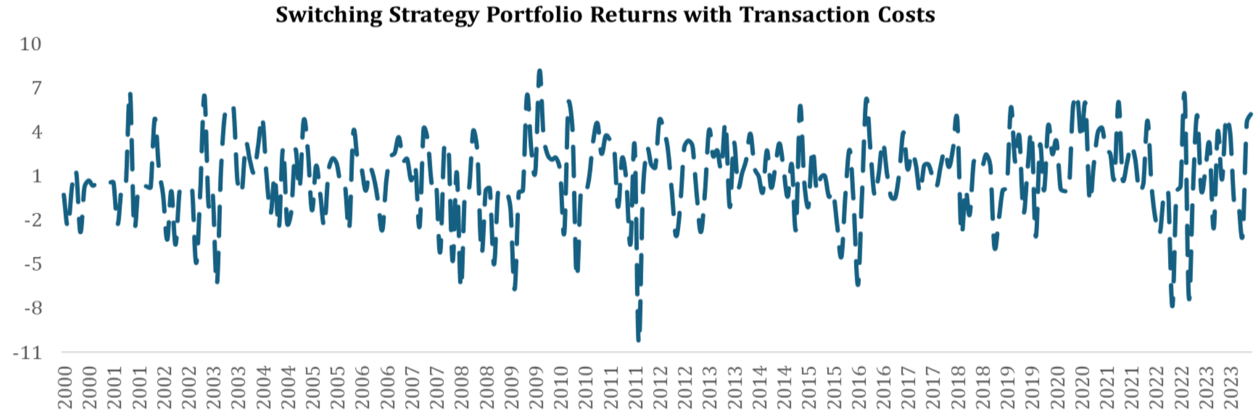
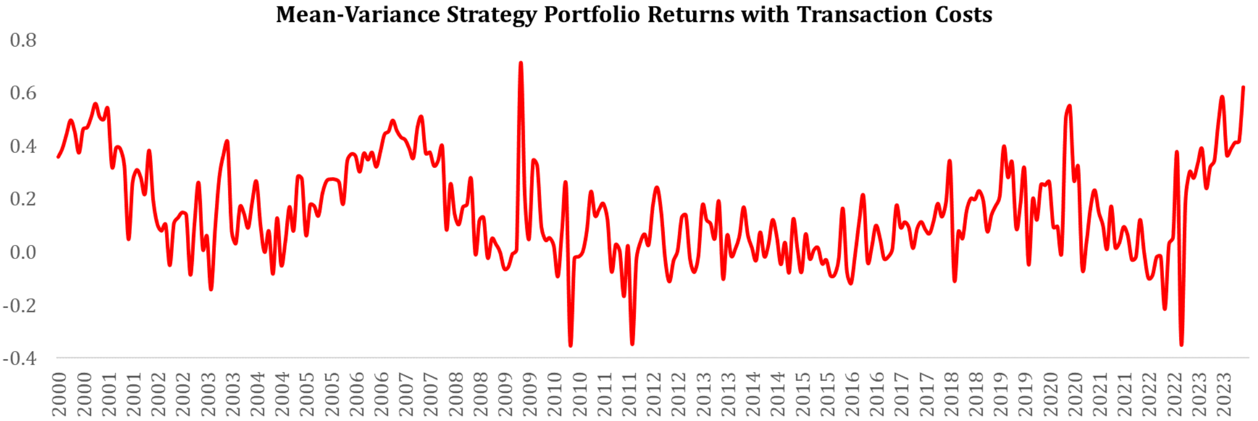
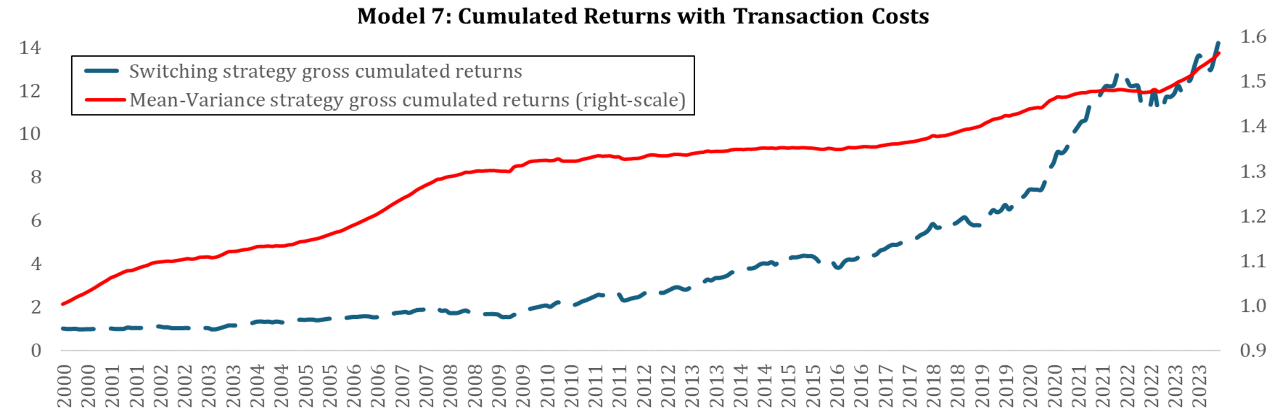


Figure 10: Portfolio Strategy Results under Model 7 (Regime Switching, Periodically Collapsing Bubbles without the Impact of the FFR on Time-Varying Probs.) for Switching and Mean-Variance Portfolio (with $\gamma = 2$) Strategies, after Transaction Costs







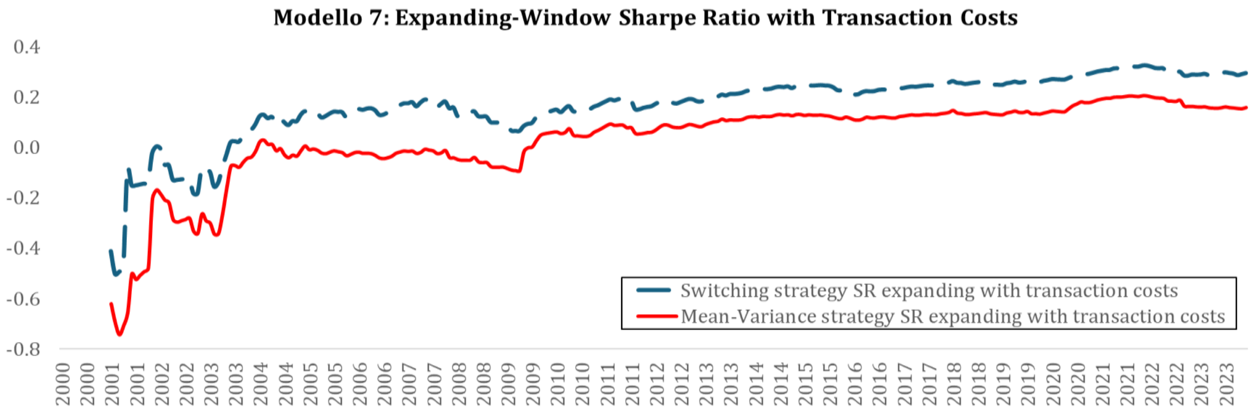
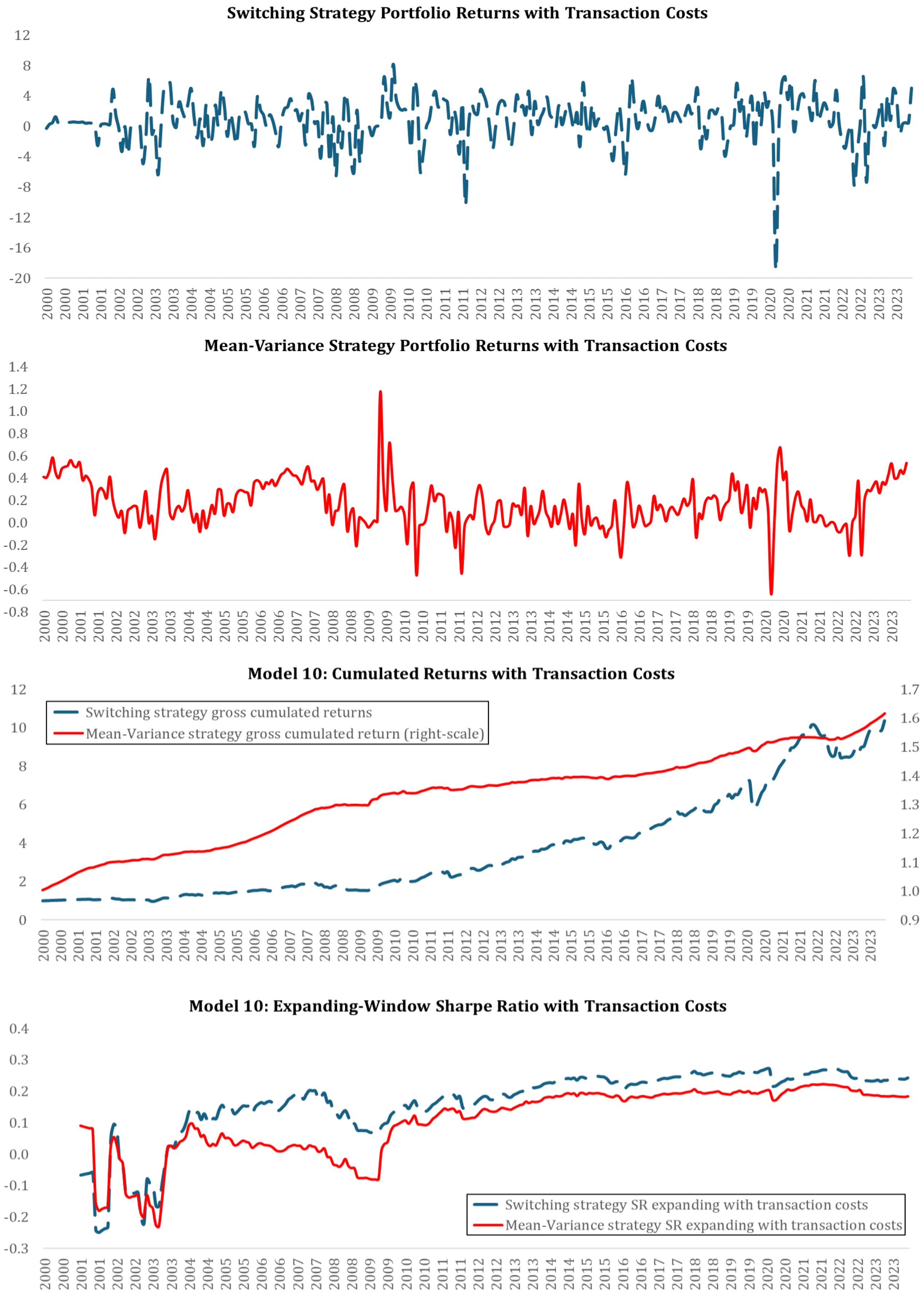


Figure 11: Portfolio Strategy Results under Model 10 (Regime Switching, Periodically Collapsing Bubble with Impact of the FFR on Time-Varying Probs.) for Switching and Mean-Variance Portfolio (with $\gamma = 2$) Strategies



Appendix A: Derivation of the Regime Switching Model

In order to linearise the model, we take the first order Taylor series approximation around an arbitrary triple b_0, sr_0, and V_0^x and arrive at a switching regression model that has a single state-independent probability of switching regimes $\left(q(b_t, sr_t, V_t^x)\right)$.

Start from the no-arbitrage relationships

$$E(r_{t+1}|S) = \left[\mu(1-b_t) + \frac{\mu}{q(b_t, sr_t, V_t^x)} b_t - \frac{1 - q(b_t, sr_t, V_t^x)}{q(b_t, sr_t, V_t^x)} u(b_0, sr_0)\right] \quad with\, prob.\, q(b_t, sr_t, V_t^x)$$

$$E(r_{t+1}|C) = [\mu(1-b_t) + u(b_t, sr_t)] \quad with\, prob. -q(b_t, sr_t, V_t^x)$$

where S and C are the survival and the collapsing regimes, respectively; $u(b_t, sr_t)$ is a continuous and everywhere differentiable function such that $u(0) = 0$ and $0 \leq \frac{\partial u(b_t, sr_t)}{\partial b_t} \leq 1$, to be interpreted as the real value of the bubble after it partially collapses; $q(b_t, sr_t, V_t^x)$ is the probability of the bubble continuing to exist, that is plausibly assumed to be a function of the absolute value of the relative size of the bubble, of the short-term interest rate, and of trading volume.

As it is well known, the (first-order) Taylor series expansion of a function $f(x, y)$ is given by:

$$f(x,y,z) = \left(f(x_0,y_0,z_0) - f_x'(x_0,y_0,z_0)x_0 - f_y'(x_0,y_0,z_0)y_0 - f_z'(x_0,y_0,z_0)z_0\right) + f_x'(x_0,y_0,z_0)x + f_y'(x_0,y_0,z_0)y + f_z'(x_0,y_0,z_0)z$$

where $f_x'(x,y,z)$ is the partial derivative of $f(x,y,z)$ with respect to x and $f_y'(x,y,z)$ and $f_z'(x,y,z)$ are defined accordingly. Note that in what follows we shall use the notation

$$\frac{\partial g(w_0)}{\partial w} \equiv \left.\frac{\partial g(w)}{\partial w}\right|_{w=w_0}.$$

Under this setting, with reference to expected returns in the bubble surviving state, the Taylor Series approximation can be derived as:

$$f(b_0, sr_0, V_0^x) = \mu(1-b_0) + \frac{\mu b_0}{q(b_0, sr_0, V_0^x)} - \frac{u(b_0, sr_0) - q(b_0, sr_0, V_0^x)u(b_0)}{q(b_0, sr_0, V_0^x)}$$

and:

$$f_b'(b_0, sr_0, V_0^x) = -\mu + \left[\frac{q(b_0, sr_0, V_0^x)\mu - \mu b_0 \frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2}\right] - \left[\frac{q(b_0, sr_0, V_0^x)\left(\frac{\partial (b_0, sr_0)}{\partial b} - \left(q(b_0, sr_0, V_0^x)\frac{\partial (b_0, sr_0)}{\partial b} + u(b_0, sr_0)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}\right)\right)}{q(b_0, sr_0, V_0^x)^2} - \frac{u(b_0, sr_0)\left(1 - q(b_0, sr_0, V_0^x)\right)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2}\right]$$

We can rewrite the second part on the right-hand side as:

$$\left[\frac{q(b_0, sr_0, V_0^x)\mu - \mu b_0 \frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2}\right] = \left[\frac{q(b_0, sr_0, V_0^x)\mu}{q(b_0, sr_0, V_0^x)^2} - \frac{\mu b_0 \frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2}\right]$$

$$= \left[\frac{\mu}{q(b_0, sr_0, V_0^x)} - \frac{\mu b_0 \frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2}\right] = \frac{\mu}{q(b_0, sr_0, V_0^x)}\left[1 - \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)} b_0\right]$$

Finally, the third part on the right-hand side can be rewritten as:

$$\left[\frac{q(b_0, sr_0, V_0^x)\left(\frac{\partial u(b_0, sr_0)}{\partial b} - \left(q(b_0, sr_0, V_0^x)\frac{\partial u(b_0)}{\partial b} + u(b_0, sr_0)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}\right)\right)}{q(b_0, sr_0, V_0^x)^2} - \frac{u(b_0, sr_0)\big(1 - q(b_0, sr_0, V_0^x)\big)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2}\right]$$

$$= \left[\frac{q(b_0, sr_0, V_0^x)\frac{\partial u(b_0, sr_0)}{\partial b} - q(b_0, sr_0, V_0^x)^2\frac{\partial u(b_0, sr_0)}{\partial b} - u(b_0, sr_0)q(b_0, sr_0, V_0^x)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, V_0^x)^2} + \frac{-\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}u(b_0, sr_0) + u(b_0, sr_0)q(b_0, sr_0, V_0^x)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2}\right]$$

$$= \left[\frac{q(b_0, sr_0, V_0^x)\frac{\partial u(b_0, sr_0)}{\partial b} - q(b_0, sr_0, V_0^x)^2\frac{\partial u(b_0, sr_0)}{\partial b} - \frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}u(b_0, sr_0)}{q(b_0, sr_0, V_0^x)^2}\right]$$

$$= \left[\frac{q(b_0, sr_0, V_0^x)\frac{\partial u(b_0, sr_0)}{\partial b} - q(b_0, sr_0, V_0^x)^2\frac{\partial u(b_0, sr_0)}{\partial b}}{q(b_0, sr_0, V_0^x)^2} - \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}u(b_0, sr_0)}{q(b_0, sr_0, V_0^x)^2}\right]$$

$$= \left[\frac{\big(1 - q(b_0, sr_0, V_0^x)\big)q(b_0, sr_0, V_0^x)}{q(b_0, sr_0, V_0^x)^2}\frac{\partial u(b_0, sr_0)}{\partial b} - \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}u(b_0, sr_0)}{q(b_0, sr_0, V_0^x)^2}\right]$$

$$= \left[-\frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2}u(b_0, sr_0) + \frac{\big(1 - q(b_0, sr_0, V_0^x)\big)}{q(b_0, sr_0, V_0^x)}\frac{\partial u(b_0, sr_0)}{\partial b}\right]$$

Substituting back into the main equation, we obtain:

$$f_b'(b_0, sr_0, V_0^x) = -\mu + \frac{\mu}{q(b_0, sr_0, V_0^x)}\left[1 - \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)} b_0\right] - \left[-\frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2}u(b_0, sr_0) + \frac{\big(1 - q(b_0, sr_0, V_0^x)\big)}{q(b_0, sr_0, V_0^x)}\frac{\partial u(b_0, sr_0)}{\partial b}\right]$$

$$= -\mu + \frac{\mu}{q(b_0, sr_0, V_0^x)}\left[1 - \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)} b_0\right] + \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2} u(b_0, sr_0) - \frac{1 - q(b_0, sr_0, V_0^x)}{q(b_0, sr_0, V_0^x)} \frac{\partial u(b_0, sr_0)}{\partial b}$$

Conversely:

$$f_V'(b_0, sr_0, V_0^x) = -\mu b_0 \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2} - \left[\frac{\left(q(b_0, sr_0, V_0^x)\left(-u(b_0, sr_0)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}\right)\right) - \left(\left(u(b_0, sr_0) - q(b_0, sr_0, V_0^x)u(b_0)\right)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}\right)}{q(b_0, sr_0, V_0^x)^2}\right]$$

$$= -\mu b_0 \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2} - \left[\frac{-u(b_0, sr_0)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2}\right]$$

$$= -\mu b_0 \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2} + \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2} u(b_0, sr_0) = [u(b_0, sr_0) - \mu b_0] \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2}$$

Similarly, through similar steps, we have that:

$$f_{sr}'(b_0, sr_0, V_0^x) = -\mu b_0 \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}}{q(b_0, sr_0, V_0^x)^2} - \left[\frac{\left(q(b_0, sr_0, V_0^x)\left(\frac{\partial u(b_0, sr_0)}{\partial sr} - u(b_0, sr_0)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}\right)\right)}{q(b_0, sr_0, V_0^x)^2} - \frac{\left(\left(u(b_0, sr_0) - q(b_0, sr_0, V_0^x)u(b_0, sr_0)\right)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}\right)}{q(b_0, sr_0, V_0^x)^2}\right]$$

$$= -\mu b_0 \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}}{q(b_0, sr_0, V_0^x)^2} - \left[\frac{q(b_0, sr_0, V_0^x)\frac{\partial u(b_0, sr_0)}{\partial sr} - u(b_0, sr_0)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}}{q(b_0, sr_0, V_0^x)^2}\right]$$

$$= [u(b_0, sr_0) - \mu b_0] \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}}{q(b_0, sr_0, V_0^x)^2} - \frac{\frac{\partial u(b_0, sr_0)}{\partial sr}}{q(b_0, sr_0, V_0^x)}$$

Recalling now that

$$f(x, y, z) = \left(f(x_0, y_0, z_0) - f_x'(x_0, y_0, z_0)x_0 - f_y'(x_0, y_0, z_0)y_0 - f_z'(x_0, y_0, z_0)z_0\right) + f_x'(x_0, y_0, z_0)x + f_y'(x_0, y_0, z_0)y + f_z'(x_0, y_0, z_0)z$$

where:

$$f(b_0, sr_0, V_0^x) = \mu(1 - b_0) + \frac{\mu b_0}{q(b_0, sr_0, V_0^x)} - \frac{u(b_0, sr_0) - q(b_0, sr_0, V_0^x)u(b_0, sr_0)}{q(b_0, sr_0, V_0^x)}$$

$$f_b'(b_0, sr_0, V_0^x) = -\mu + \frac{\mu}{q(b_0, sr_0, V_0^x)}\left[1 - \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)} b_0\right] + \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2} u(b_0, sr_0) - \frac{1 - q(b_0, sr_0, V_0^x)}{q(b_0, sr_0, V_0^x)} \frac{\partial u(b_0, sr_0)}{\partial b}$$

$$f_V'(b_0, sr_0, V_0^x) = [u(b_0, sr_0) - \mu b_0] \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2}$$

$$f_{sr}'(b_0, sr_0, V_0^x) = [u(b_0, sr_0) - \mu b_0] \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}}{q(b_0, sr_0, V_0^x)^2} - \frac{\frac{\partial u(b_0, sr_0)}{\partial sr}}{q(b_0, sr_0, V_0^x)}$$

Renaming the coefficients and substituting, we can rewrite the expression for conditionally expected returns as:

$$E(r_{t+1}|S) = \beta_{S,0} + \beta_{S,b} b_t + \beta_{S,sr} sr_t + \beta_{S,V} V_t^x$$

where:

$$\begin{aligned}
\beta_{S,0} = \Bigg[& \left[\mu(1 - b_0) + \frac{\mu b_0}{q(b_0, sr_0, V_0^x)} - \frac{u(b_0, sr_0) - q(b_0, sr_0, V_0^x) u(b_0)}{q(b_0, sr_0, V_0^x)}\right] \\
& - \Bigg[-\mu + \frac{\mu}{q(b_0, sr_0, V_0^x)}\left[1 - \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)} b_0\right] + \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2} u(b_0, sr_0) \\
& - \frac{1 - q(b_0, sr_0, V_0^x)}{q(b_0, sr_0, V_0^x)} \frac{\partial u(b_0, sr_0)}{\partial b}\Bigg] b_0 - (u(b_0, sr_0) - \mu b_0) \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2} V_0^x \\
& - (u(b_0, sr_0) - \mu b_0) \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}}{q(b_0, sr_0, V_0^x)^2} sr_0 \Bigg] \\
= & \left[\mu + \frac{1 - q(b_0, sr_0, V_0^x)}{q(b_0, sr_0, V_0^x)}\left[\frac{\partial u(b_0, sr_0)}{\partial b} b_0 - u(b_0, sr_0)\right]\right. \\
& + \left[\frac{\mu}{q(b_0, sr_0, V_0^x)} b_0 - u(b_0, sr_0)\right] \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)} b_0 \\
& \left. + (u(b_0, sr_0) - \mu b_0) \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2} V_0^x + (u(b_0, sr_0) - \mu b_0) \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}}{q(b_0, sr_0, V_0^x)^2} sr_0 \right]
\end{aligned}$$

$$\beta_{S,b} = -\mu + \frac{\mu}{q(b_0, sr_0, V_0^x)}\left[1 - \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)} b_0\right] + \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2} u(b_0, sr_0) - \frac{1 - q(b_0, sr_0, V_0^x)}{q(b_0, sr_0, V_0^x)} \frac{\partial u(b_0, sr_0)}{\partial b}$$

$$\beta_{S,sr} = [u(b_0, sr_0) - \mu b_0] \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}}{q(b_0, sr_0, V_0^x)^2} - \frac{\frac{\partial u(b_0, sr_0)}{\partial sr}}{q(b_0, sr_0, V_0^x)}$$

$$\beta_{S,V} = [u(b_0, sr_0) - \mu b_0] \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2}$$

In the same way, we can derive the Taylor series approximation of the returns in the collapsing regime, when

$$f(b_0, sr_0, V_0^x) = \mu(1 - b_0) + u(b_0, sr_0)$$

$$f_b'(b_0, sr_0, V_0^x) = -\mu + \frac{\partial u(b_0, sr_0)}{\partial b}$$

$$f_{sr}'(b_0, sr_0, V_0^x) = -\frac{\partial u(b_0, sr_0)}{\partial sr}$$

$$f_V'(b_0, sr_0, V_0^x) = 0$$

Moreover, we can show that

$$E(r_{t+1}|C) = \beta_{C,0} + \beta_{C,b} b_t + \beta_{C,sr} b_t$$

where:

$$\beta_{C,0} = \mu + u(b_0, sr_0) - \frac{\partial u(b_0, sr_0)}{\partial b} b_0$$

$$\beta_{C,b} = -\mu + \frac{\partial u(b_0, sr_0)}{\partial b}$$

$$\beta_{C,sr} = \frac{\partial u(b_0, sr_0)}{\partial b}$$

Since we have defined the probability of survival as:

$$Pr(S) = \ell\left(\beta_{q,0} + \beta_{q,babs}|b_t| + \beta_{q,bsqr} b_t^2 + \psi_{q,srabs}|sr| + \beta_{q,srsqr} sr_t^2 + \gamma_{q,V} V_t^x\right)$$

the full linearised model is:

$$E(r_{t+1}|W_{t+1} = S) = \beta_{S,0} + \beta_{S,b} b_t + \beta_{S,sr} sr_t + \beta_{S,V} V_t^x$$

$$E(r_{t+1}|W_{t+1} = C) = \beta_{C,0} + \beta_{C,b} b_t + \beta_{C,sr} sr_t$$

$$Pr(S) = \ell\left(\beta_{q,0} + \beta_{q,babs}|b_t| + \beta_{q,bsqr} b_t^2 + \psi_{q,srabs}|sr| + \beta_{q,srsqr} sr_t^2 + \gamma_{q,V} V_t^x\right).$$

In our empirical application, we shall also include the term $\beta_{C,V} V_t^x$ in the regime-dependent regression in the collapsing state, thus allowing the data to reject its presence. Its formal inclusion would require to also make the $u(\cdot)$ function depend on V_t^x, a fact that we do not find implausible but that we refrain from imposing as a matter of logical necessity.

Appendix B: Derivation of the Restrictions on the Coefficients of the Switching Regression Model

From the model described in the main text and in Appendix A, we can derive certain conditions that must hold if a speculative bubble is present in stock prices, as always under the restriction that the bubble component follows the process assumed.

Firstly, we know that the coefficients $\beta_{q,b}$ and $\gamma_{q,V}$ must be negative by construction if the speculative bubble model theory is correct. Secondly, the coefficient on the bubble component in the collapsing regime should have a negative sign. This is because, from the derivation of the augmented regression model, we known that:

$$\frac{\partial E_t(r_{t+1}|C)}{\partial b_t} = -\mu + \frac{\partial u(b_t, sr_t)}{\partial b_t} - \frac{\partial u(b_0, sr_t)}{\partial sr_t}.$$

Remember that under a standard Gordon's growth model, it must be:

$$\mu = \frac{(1+\rho)e^{a_0}}{\rho} = \left(1 + \frac{1}{\rho}\right) e^{a_0} > 1,$$

where ρ is the long-run price-dividend ratio. Since dividends tend to grow rather than shrink over time, a_0 should be positive and thus $-\mu$ should be smaller than -1. This is the logical assumption for us to make as stocks must have a positive expected fundamental return for a rational investor to hold equity. However, because the bubble must shrink in the collapsing regime, we know that:

$$0 \le \frac{\partial u(b_t, sr_t)}{\partial b_t} \le 1.$$

This implies that the second term in $\frac{\partial E_t(r_{t+1}|C)}{\partial b_t} = -\mu + \frac{\partial u(b_t)}{\partial b_t}$ is smaller than 1 and thus the coefficient $\beta_{C,b}$ must be smaller than $-\frac{\partial u(b_0, sr_0)}{\partial sr_t}$. If we assume that $\frac{\partial u(b_0, sr_0)}{\partial sr_t} > 0$, i.e., higher rates tend to make a bubble collapse more likely and frequent (i.e., the duration of the bubbles is lower) but their collapse less pronounced, then this yields

$$\frac{\partial E_t(r_{t+1}|C)}{\partial b_t} = -\mu + \frac{\partial u(b_t, sr_t)}{\partial b_t} - \frac{\partial u(b_t, sr_t)}{\partial sr_t} < -\frac{\partial u(b_t, sr_t)}{\partial sr_t} < 0.$$

In the same way, we can derive the condition that $\beta_{S,b} > \beta_{C,b}$:

$$\begin{aligned}
\beta_{S,b} &= \frac{\partial E_t(r_{t+1}|S)}{\partial b_t} \\
&= -\mu + \frac{\mu}{q(b_0, sr_0, V_0^x)}\left[1 - \frac{sign(b_0)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)} b_0\right] \\
&\quad + \frac{sign(b_0)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2} u(b_0, sr_0) - \frac{1 - q(b_0, sr_0, V_0^x)}{q(b_0, sr_0, V_0^x)} \frac{\partial u(b_0, sr_0)}{\partial b} \\
&= -\mu + \frac{\partial u(b_0, sr_0)}{\partial b} - \frac{\frac{\partial u(b_0, sr_0)}{\partial b}}{q(b_0, sr_0, V_0^x)} + \frac{\mu}{q(b_0, sr_0, V_0^x)} - \mu \frac{sign(b_0)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b} b_0}{q(b_0, sr_0, V_0^x)^2} \\
&\quad + \frac{sign(b_0)\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)^2} u(b_0, sr_0)
\end{aligned}$$

But we know that:

$$\beta_{C,b}=\frac{\partial E_t(r_{t+1}|C)}{\partial b_t}=-\mu+\frac{\partial u(b_t,sr_t)}{\partial b_t}-\frac{\partial u(b_t,sr_t)}{\partial sr_t}$$

which must also hold for the special case $(b_t, sr_t) = (b_0, sr_o)$. Thus substituting, rearranging and simplifying:

$$\beta_{S,b}=\frac{\partial E_t(r_{t+1}|C)}{\partial b_t}$$
$$=\beta_{C,b}+\frac{\partial u(b_t,sr_t)}{\partial sr}+\frac{\mu-\frac{\partial u(b_0,sr_0)}{\partial b}}{q(b_0,sr_0,V_0^x)}-\mu\frac{sign(b_0)\frac{\partial q(b_0,sr_0,V_0^x)}{\partial b}b_0}{q(b_0,sr_0,V_0^x)^2}$$
$$+\frac{sign(b_0)\frac{\partial q(b_0,sr_0,V_0^x)}{\partial b}}{q(b_0,sr_0,V_0^x)^2}u(b_0,sr_0)$$

The third term is positive and so, when $sign(b_0) = +1$, i.e., as long as the bubble is positive, is the fourth term since:

$$\frac{\partial q(b_0,sr_0,V_0^x)}{\partial b}<0$$

The fifth term is negative but smaller in absolute value than the sum of the third and the fourth terms as

$$[u(b_0,sr_0)-\mu]\frac{sign(b_0)\frac{\partial q(b_0,sr_0,V_0^x)}{\partial b}b_0}{q(b_0,sr_0,V_0^x)^2}>0$$

Therefore $\frac{\partial u(b_t,sr_t)}{\partial sr_t} > 0$ turns out to be sufficient for $\beta_{S,B} > \beta_{C,B}$, however, we cannot say anything about the sign of this coefficient since it depends on relative size of all the parts. However, when the bubble is negative, $\beta_{S,B} \leq \beta_{C,B}$ becomes possible.

Finally, we show that:

$$\beta_{S,V}=[u(b_0,sr_0)-\mu b_0]\frac{\frac{\partial q(b_0,sr_0,V_0^x)}{\partial V^x}}{q(b_0,sr_0,V_0^x)^2}$$

The term outside brackets of this equation is negative since:

$$\frac{\partial q(b_0,sr_0,V_0^x)}{\partial V^x}<0$$

Furthermore, the term inside brackets is also negative since $u(b_0, sr_0) \leq b_0$ and we know that μ is greater than 1. Therefore, the product of the two terms is positive and thus $\beta_{S,V}$ is always positive.

$$\beta_{S,0} = \left[\mu + \frac{1 - q(b_0, sr_0, V_0^x)}{q(b_0, sr_0, V_0^x)}\left[\frac{\partial u(b_0, sr_0)}{\partial b} b_0 - u(b_0, sr_0)\right]\right.$$
$$+ \left[\frac{\mu}{q(b_0, sr_0, V_0^x)} b_0 - u(b_0, sr_0)\right] \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial b}}{q(b_0, sr_0, V_0^x)} b_0$$
$$\left. + (u(b_0, sr_0) - \mu b_0) \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial V^x}}{q(b_0, sr_0, V_0^x)^2} V_0^x + (u(b_0, sr_0) - \mu b_0) \frac{\frac{\partial q(b_0, sr_0, V_0^x)}{\partial sr}}{q(b_0, sr_0, V_0^x)^2} sr_0\right]$$

and

$$\beta_{C,0} = \mu + u(b_0, sr_0) - \frac{\partial u(b_0, sr_0)}{\partial b} b_0$$

which are clearly different.