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Bubbles and Persuasion with Second Order Uncertainty

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Abstract

Recent empirical studies suggest that, during times of unexpected innovation, agents heterogeneously update their beliefs about an asset fundamental value, and they are uncertain about other agents' beliefs on it. In this paper I show that, when there is uncertainty about the *market sentiment*, defined as other investors' beliefs over an asset fundamental value, market manipulation can act through a previously unconsidered channel, by misleading agents' learning on the market sentiment. This novel type of market manipulation becomes a severe concern with the recent diffusion of big data on agents' beliefs, as it could strengthen existing financial bubbles, or even give rise to new ones.

JEL Classification: C73, D82, D83, D84, G14, G24

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1 Introduction

When it comes to controlling human beings, there is no better instrument than lies. Because you see, humans live by beliefs. And beliefs can be manipulated. The power to manipulate beliefs is the only thing that counts.
Michael Ende, *The Neverending Story*

Recent empirical studies suggest that, during times of financial turbulence, investors try to gauge, independently of their own beliefs and often incorrectly, other investors' beliefs over the market's fundamentals, in order to predict their strategies (Egan, Merkle and Weber 2014; De Martino, O'Doherty, Ray, Bossaerts and Camerer 2013). If this is the case, agents' decisions, and thus prices, are driven not only by their belief about the asset fundamental value, but also by their beliefs on other agents' beliefs about it, which I will refer to as their *second order beliefs*. In particular, the more optimistic second order beliefs are, the higher the price of the asset will be, and the more it could diverge from the fundamental value. As a consequence, when *second order uncertainty*, that is, uncertainty over other agents' beliefs, is in play, any information agents receive on the *market sentiment*, here defined as investors' beliefs over an asset fundamental value, influences their investing behavior and, thus, prices. This gives rise to a new concern, as it could be profitable for a financial company with an interest in a mispricing, and the possibility of aggregating information on the market sentiment, to choose the information aggregation structure to manipulate agents' second order beliefs, without influencing their beliefs over fundamentals. This way, investors are induced to stay on average longer invested in the market, giving potentially rise to a financial bubble. How diffuse this type of market manipulation can be and who can have access to the aggregation of information about market sentiment? Indeed, in financial markets, there is a growing attention to data about market sentiment. In the past few years, with the arrival of big data on investors' trading intentions and decisions, we have observed a rising interest for social media as a vehicle of information about the market sentiment. Hedge funds and large investors have started to buy and use measures of market sentiment derived from tweets and other social media instruments in order to make trading decisions. Twitter even signed an agreement with International Business Machines Corp, granting the computer company access to the full “fire hose”of tweets sent from around the world every day. Thus, with the arrival of big data, the risk of manipulation of second order beliefs is getting higher, and should be source of concern for regulators. In the financial literature, there are many empirical and theoretical works that show the conflict of interest of firms that provide information on assets fundamentals to investors, but it lacks an analysis of this new and different type of information, that allows to learn what other investors think about an asset. This paper wants to fill that gap. Concerning who could manipulate such information, there are mainly two types of financial companies that process, transform and transfer to investors information about market sentiment: analyst firms and big investment and intermediary firms. On one hand, new analyst companies, like iSentium LLC and TheySay, emerged, which aggregate the market sentiment information from the social media. On the other hand, large investment banks that serve as intermediaries, like TD Ameritrade and Merril Lynch, have now access to and process for their clients a lot of information on market sentiment[1]. These companies could find it optimal to sustain long periods of mispricing, as this leads to higher commissions. For instance, they could find it profitable to create an information aggregation structure such that market sentiment indicators derived from it are relatively less precise when the true market sentiment is negative than when it is positive. This way, they would be able to induce investors to stay on average longer invested in the market, and to give rise to or strengthen financial bubbles. If this is the case, we expect market sentiment indicators created by such companies to be more powerful in predicting a negative market sentiment than a positive one. In this paper, I show theoretically how these companies may optimally manipulate agents’ second order beliefs, to ultimately influence their strategies, and, thus, the price of an asset. In particular, I show how such manipulation could fuel an already existing financial bubble, or even start a new one.

In order to study this potentially harmful threat, I model a financial market for an asset where, after an unexpected innovation, both the fundamental value and the

[1]See, for instance, *The Wall Street Journal*, July 6th 2015, “Tapping Twitter to Gauge Stock Sentiment.”

price of the asset start growing at a rate higher than before. Moreover, in each period, there is a positive probability of a mispricing, i.e., that the fundamental value, but not the price, stops growing at such a high rate. Building on the model of Abreu and Brunnermeier (2003, hereafter AB), as soon as the mispricing starts, agents sequentially and privately receive the information that the mispricing has started, but not when it started. In order to incorporate the recent empirical findings on the independence of beliefs over the asset fundamental value from second order beliefs, I enrich the AB model assuming that agents have heterogenous prior beliefs on when a mispricing could potentially arise, and are also uncertain about the prior beliefs of other agents in the market. This is the same of assuming heterogeneity in prior beliefs about the fundamental value of an asset, and uncertainty about other agents' beliefs on it. In particular, I assume that the market is populated by two types of agents, *optimists* and *pessimists*, and that pessimists are uncertain about the proportion of agents of the same type. Both optimists and pessimists are fully invested in the market and characterized by different prior beliefs on the starting point of the mispricing. An agent is an optimist (pessimist) if he ex ante believes that in every period it is less (more) probable than for a pessimist (optimist) that a mispricing starts. The intensity of the market sentiment is defined as the proportion of optimists invested in the market. Pessimistic agents, being uncertain about the proportion of optimists in the market, are uncertain about the market sentiment. To keep things tractable, I assume that there are only two possible compositions of the market. In the first one, there are only pessimists. In this case, the market is fully pessimistic and there is a *low* market sentiment. In the other case, there is also a positive fraction of optimists, and, thus, the market is optimistic, and there is a *high* market sentiment. Pessimistic agents, being uncertain about the prevailing market sentiment, assign a positive probability to each one. Every pessimistic agent gives prior probability $\alpha_0 \in (0, 1)$ to the first composition, and probability $1 - \alpha_0$ to the second one, and this is common knowledge. The two assumptions of heterogeneity in beliefs and uncertainty on others' beliefs incorporate the results of recent empirical studies on the expectations of agents during times of abnormal financial markets. For instance, Egan, Merkle and Weber (2014) conduct a panel survey of online self-directed investors from September 2008 to December 2009, to collect agents' return expectations and their beliefs about other investors' expectations. They find heterogeneity in return expectations, and heterogeneity and inaccuracy in predicting others' expectations. More generally, theoretical models with second and higher order uncertainty seem more helpful than traditional rational expectations models to explain some empirical evidence, such as the high trading volumes seen during financial bubbles and the excess volatility puzzle in asset pricing[2]. Moreover, from a theoretical point of view, explicitly modelling second order uncertainty allows to investigate the previously not considered channel of *second order manipulation*, that is manipulation of second order beliefs. Indeed, as it will be clearer later, this type of manipulation is possible only when agents are uncertain about other agents' beliefs about the fundamentals. To study the manipulation threat, I assume that, as soon as the game starts, a financial company, with access to information on the market sentiment, sends to investors a public signal on the true market sentiment. This will make pessimistic agents update their prior on the true composition of the market to a posterior α_1. The aim of the financial company is to maximize its utility. In order to model the incentives of the company, I use the Bayesian persuasion mechanism introduced by Kamenica and Gentzkow (2011, hereon KG), for which the company cannot change the information once aggregated, but it can choose the precision of the public signals. In this environment, agents must choose when to sell the asset, considering that, if they sell it too late, the mispricing could have already been corrected by that time, and they will get a lower price. This could happen for two reasons: either because enough agents have already sold the asset, bringing the price back to the fundamental value, or because of an external event. As in AB, a mispricing becomes a *bubble* when there are enough agents informed about the mispricing, and still invested in the market, who would be able, by selling the asset, to bring back the price to the fundamental value.

Under these assumptions, I investigate the equilibria of the model and ask whether it

[2]For theoretical results on the high trading volumes and volatility predicted by models with heterogenous and higher order beliefs, see Atmaz and Basak (2018), Scheinkman and Xiong (2003), Harris and Raviv (1993), Townsend (1983) and Basak (2000). For an empirical test of the predictive power of these models, see, for instance, Ofek and Richardson (2003), Chen, Hong, and Stein (2001) and Monnin (2004).

is possible and profitable for the financial company to manipulate second order beliefs and, in doing so, to fuel financial bubbles. As I will show, the lack of common knowledge on who is the first agent to receive the information on the mispricing, together with the uncertainty about the prevailing market sentiment, can lead agents to find it optimal to ride the bubble for a while. In particular, if α_1, the updated probability pessimistic agents assign to a low market sentiment, is higher than a certain threshold $\overline{\alpha}$, there is a unique equilibrium where the bubble does not even start when the market sentiment turns out to be low, while it starts but it bursts soon if the market sentiment is high. In this case, pessimistic agents behave as if they were certain of the market being populated only by pessimists, and exit immediately the market. The reason why pessimists exit immediately the market is that, as soon as they are informed of the mispricing, their perceived risk of an immediate crash becomes too high to find it optimal to remain invested. On the other hand, if α_1 is lower than the same threshold, there is a unique equilibrium where agents' strategies make the bubble grow for a while if the world is fully pessimistic, and make it burst very late if the world is populated also by some optimists. The intuition is that, in this case, pessimistic agents give high enough probability to the market being populated also by optimists, who are believed to stay longer invested in the market. Differently from the case where α_1 is high, pessimists are now confident that optimists will sustain the bubble for a while and, thus, their perceived risk of an immediate crash is much lower. This makes them willing to ride the bubble for some time. Thus, even if it turns out that the market is populated only by pessimistic agents, the bubble will inflate for a while, in contrast with what happens in a world where agents are sure of being all pessimists. This means that there are cases in which the bubble is self-fulfilling, in the sense that a bubble exists in a market populated only by pessimists, based on their initial belief that there are some optimistic agents that sustain the bubble. This belief makes pessimists more optimistic about the duration of a bubble, and thus makes them willing to stay invested for a while, giving themselves rise to a bubble. Moreover, when the initial probability α_0 is high enough, a financial information company finds it profitable to send a manipulated public signal to induce pessimistic agents to update α_0 to a lower value α_1 and to stay longer invested in the market. This will make also optimistic agents to stay longer invested in the market. This behavior, on average, makes financial bubbles last longer and happen more frequently.

Thus, this paper shows that when there is second order uncertainty, there is a novel channel, which I refer to as a *second-order beliefs* channel, through which someone could manipulate the market. The key feature of this type of manipulation is that the manipulator misleads investors' second-order beliefs to induce them to believe that the market is more optimistic than in reality. In other words, the manipulator is able to move investors' second order beliefs on fundamentals in its preferred direction, while holding their beliefs on fundamentals fixed. These manipulated beliefs make agents willing to ride the bubble for a while, and, consequently, a new bubble starts or the bubble crashes later than in cases of no manipulation. Interestingly, this channel of manipulation does not work when there is common knowledge of beliefs, because agents already know the beliefs of others. Instead, with uncertainty of the market sentiment, agents' strategies depend both on their beliefs on the fundamental value and on their second order beliefs. This means that manipulation can work through both channels: the traditionally known channel of manipulation of information about fundamentals and the novel channel of manipulation of information about other agents' beliefs. To the best of my knowledge, this paper is the first to highlight this latter channel of market manipulation and to explore its consequences.

Related Literature

This paper studies market manipulation in the context of financial bubbles with heterogenous beliefs and lack of common knowledge on other agents' beliefs. Harrison and Kreps (1978) introduce heterogenous beliefs among traders and show that overpricing is possible. Guth (1989) is the first to relax the assumption of common knowledge in financial markets. Kraus and Smith (1998) deals with uncertainty about the beliefs of others in a standard security market and show that the price of the risky asset is higher than the fundamental value at all dates. Similarly, Allen, Morris and Postlewaite (1993) and Conlon (2004) show that the lack of common knowledge of common priors drives stock prices and lets bubbles survive in equilibrium. Morris, Postlewaite and Shin (1995), Allen, Morris and Shin (2006) and Banerjee, Kaniel and Kremer (2009) show that for a price drift to exist it is necessary to have higher order uncertainty, on top of agreed differences of opinions. My paper shares with all these models the idea that, if there is lack of common knowledge on the beliefs of others, bubbles can exist in equilibrium. At the same time, differently from these models, it studies this in the context of a market with many agents. My model builds on Abreu and Brunnermeier (2003), and Doblas Madrid (2012), who, in the context of a market, assume that there is lack of common knowledge on when agents receive their private information, while assuming common knowledge of common priors of the fundamental value. This assumption, while introducing heterogeneity in agents' beliefs and lack of common knowledge of others' beliefs on fundamentals, makes agents' second and higher order beliefs be perfectly correlated with their beliefs over the asset fundamental value. To avoid perfect correlation, and to incorporate the realistic possibility that agents with the same prior beliefs have different second order beliefs, I assume heterogenous prior and lack of common knowledge of others' priors. With this beliefs structure, manipulation of second order beliefs can be isolated and studied independently of manipulation of beliefs over the asset fundamental value.

Concerning the literature on stock price manipulation, according to Allen and Gale (1992), stock price manipulation can be divided into three categories: trade-based, action-based, and information-based manipulation. Allen and Gale (1992) study a model of trade-based stock price manipulation with uninformd agents. Chakraborty and Yilmaz (2004) show that if the market faces uncertainty about the existence of informed traders and there are a large number of trading periods, long-lived informed traders will manipulate in every equilibrium. Vila (1989) and Bagnoli and Lipman (1990) consider the possibility of profitable action-based manipulation. Vila (1989), Benabou and Laroque (1992) and Van Bommel (2003) look at information-based manipulation. My paper falls in the information-based manipulation literature and shares with these models the idea that an agent profitably use some information to manipulate the market. Differently from these models, where insiders have some information related to the fundamental value of the asset, my paper considers a different type of information: information on market sentiment.

Finally, this paper also contributes to the recent literature on Bayesian persuasion. Kamenica and Gentkow (2011) introduce the idea of Bayesian persuasion to show that agents can be induced by a manipulator to update their beliefs in his preferred direction, without violating Bayes' rule. A recent literature has applied this idea to regime-change games, as in Goldstein and Huang (2016), to political elections, as in Alonso and Camara (2016), and to credit markets, as in Faria e-Castro, Martinez and Philippon (2016). In this paper, I apply this idea to a model of financial bubbles with second order uncertainty.

The paper is structured as follows. Next section describes the model. Section 3 contains the equilibrium analysis when second order manipulation is in play. Finally, section 4 concludes.

2 The Model

Consider a market for an asset which, at time $t = 0$, is populated by a mass 1 of rational agents, each owning 1 unit of it. Until time $t = 0$, the price and the fundamental value of the asset grow at the risk free interest rate r. At time $t = 0$, there is an unexpected positive shock that makes the fundamental value and the price grow at a rate $g > r$, until a random time t_0. From time t_0 on, the fundamental value and the price of the asset diverge, as in t_0 the fundamental value falls down to a lower level and, from there on, it keeps growing at such lower level. Thus, from time t_0 on, there is a *mispricing* in the market. The price continues to diverge from the fundamental value either 1) until a mass k of rational agents sell out, making the price fall back at the fundamental value, or 2) until $t_0 + \bar{\tau}$ is reached, whatever happens first. Here, $\bar{\tau}$ represents the maximum number of periods that a mispricing lasts before an exogenous event makes the price fall down to coincide again with the fundamental value of the stock. For simplicity the price and the fundamental value at $t = 0$ are normalized to 1. Formally, starting from $t = 0$, and until the mispricing ends, the price is described by the function $p(t) = e^{gt}$, and this coincides with the fundamental value until t_0. After t_0, and before the end of the mispricing, the price is the sum of two components: βe^{gt}, the bubble component of the price, while $(1-\beta)$ is the fraction of the price justified by the fundamentals. Finally, when the mispricing periods end, the price falls down at the level $(1-\beta)e^{gt}$ and, thus, it coincides again with the fundamental value of the asset.

If the price falls because of the selling pressure of a mass k of agents, then the burst is said to happen for *endogenous* reasons. Indeed, during the mispricing, the price is assumed to be kept higher than the fundamental value by some unmodelled behavioral traders, who believe that the price will keep growing at the higher rate forever. Only when at least a mass k of rational agents are out of the market, the demand by behavioral traders cannot absorb the increased supply of the asset, and, thus, the price reveals the sales. In case the price falls at time $t_0+\bar{\tau}$, instead, the burst is said to happen for *exogenous* reasons.

The information structure is the following. Before the unexpected shock that happens at time $t=0$, and unmodelled here, agents agree on the fundamental value of the asset. This can be seen as the steady state of a learning process where agents, who started with different priors and received new information, updated their beliefs in a Bayesian fashion ending up agreeing on their posteriors about the value of the asset. At time $t=0$, an unexpected positive shock makes agents disagree once again about the fundamental value, and heterogenous beliefs come newly into place. The heterogeneity is captured by introducing two types of agents who differ in the ex-ante probability they assign to the random time t_0: the h-types, or *pessimists*, and the l-types, or *optimists*. Both types believe that t_0 is exponentially distributed on $[0,\infty)$. An h-type believes that the cumulative distribution function is $\Phi^h(t_0)=1-e^{-\lambda^h t_0}$, while an l-type believes that the cumulative distribution function is $\Phi^l(t_0)=1-e^{-\lambda^l t_0}$, with $\lambda^h>\lambda^l$. This can be interpreted as the h-types assigning in every period a higher probability than the l-types to the event that a mispricing has started. Moreover, starting from t_0, agents become sequentially and privately informed of the mispricing. In every period $t\geq t_0$ a mass $1/\eta$ of agents becomes informed that a mispricing has started, but not of how many agents already received the information. Since by $t=t_0+\eta k$ enough agents are informed of the current mispricing to be able to correct the price, any mispricing after this point will be called a *bubble*. Type (h,t_i) (resp. (l,t_i)) denotes an h (resp. l) type who became aware of the mispricing at time t_i, where $t_i\in[t_0,t_0+\eta]$, and started with priors $\Phi^h(t_0)=1-e^{-\lambda^h t_0}$ (resp. $\Phi^l(t_0)=1-e^{-\lambda^l t_0}$). When an agent receives the private information, he updates his belief about the starting point of the mispricing, according to Bayes' rule. From the point of view of agent t_i, no matter his type, h or l, the support for t_0 becomes $[t_i - \eta, t_i]$ and the corresponding posterior distribution is:

$$\Phi^h(t_0 \mid t_i) = \frac{e^{\lambda^h \eta} - e^{\lambda^h (t_i - t_0)}}{e^{\lambda^h \eta} - 1}$$

if he is an h-type, and:

$$\Phi^l(t_0 \mid t_i) = \frac{e^{\lambda^l \eta} - e^{\lambda^l (t_i - t_0)}}{e^{\lambda^l \eta} - 1}$$

if he is an l-type. Thus, an h-type, who received his private information in the same period of an l-type, is less confident than the l-type of being an early informed agent. In other words, having the same support for t_0, the h-type assigns a higher probability to the event that the mispricing has started long before t_i. Thus, a pessimist who received the private information in the same date of an optimist, assigns higher probability to having being an uninformed agent in a bubble economy. To model second order uncertainty, I assume that there are two possible compositions of the market, or states of the world ω. In one case, there is a totality of pessimistic agents, and I refer to this state of the world with the letter P, to indicate a pessimistic market. In the other case, there is a positive fraction $(1 - \rho) < 1$ of optimistic agents, and I refer to this state of the world with the letter O, to indicate an optimistic market. Optimistic agents know they are in the optimistic market. On the other hand, pessimistic agents are uncertain about the true composition of the market. They believe that there are only h-types, with prior probability $\alpha_0 \in (0, 1)$, and that there is a proportion $\rho \in (0, 1)$ of h-types, with prior probability $1 - \alpha_0$. Moreover, to model second order manipulation, I introduce a financial company that, at time $t = 0$, sends a public signal on the true composition of the market. I assume the utility function of the company, $u(\cdot)$, is increasing and concave (or linear) in the length τ of the bubble, everywhere nonnegative, with $u(\tau = 0) = 0$ and $u(\tau = \bar{\tau} - \eta k) = K_1$. Thus, the longer the bubble lasts, the higher the utility for the financial company is. The company is uncertain about the true market sentiment, and shares with pessimistic investors the same prior belief α_0. Before sending the signal, the company is able to aggregate information on agents' beliefs about the fundamental value of an asset. In the spirit of KG model of Bayesian persuasion, the company cannot change the information once aggregated, but it can choose the precision of signals S about optimism and pessimism in the market. Since there are only two states of the world, it follows from KG model that the analysis can be restricted to two straightforward signals: a signal $S = P$ for a pessimistic market and a signal $S = O$ for an optimistic one. Thus, the company chooses $pr(S = P \mid \omega = O)$ and $pr(S = P \mid \omega = P)$, in order to maximize his utility. Once the company has chosen the signals precision, agents receive the public signal. Investors see the signal precision chosen by the company, $pr(S = P \mid \omega = O)$ and $pr(S = P \mid \omega = P)$, and, once received the signal, they update their beliefs over the distribution of optimism in the market, using Bayes' rule. Thus, posterior beliefs are:

$$\alpha_1(\omega = P \mid s = P)) = \frac{pr(S = P \mid \omega = P)\alpha_0}{pr(S = P \mid \omega = P)\alpha_0 + pr(S = P \mid \omega = O)(1 - \alpha_0)}$$

and

$$\alpha_1(\omega = P \mid s = O)) = \frac{pr(S = O \mid \omega = P)\alpha_0}{pr(S = O \mid \omega = P)\alpha_0 + pr(S = O \mid \omega = O)(1 - \alpha_0)}.$$

In this environment, an agent must choose when to sell the asset, with the restriction that he cannot reenter once exited. Equivalently, in every period until when he exits, an agent chooses whether to stay or leave the stock market. I assume, for ease of notation and without loss of generality, that agents can only sell the entire asset, and not fractions of it. For every t, $\sigma^h(t, t_i)$ and $\sigma^l(t, t_i)$ denote the selling pressure at time t of a type (h, t_i) and (l, t_i) respectively. In particular, $\sigma^{(\cdot)}(t, t_i) = 0$ means that the agent is still invested in the market at time t, while $\sigma^{(\cdot)}(t, t_i) = 1$ means that the agent is out of the market at time t. A strategy of an agent (h, t_i) can be viewed as the mapping $\sigma^h(\cdot, t_i) : [0, t_i + \bar{\tau}] \longrightarrow [0, 1]$. A strategy of an agent (l, t_i) is given by the mapping $\sigma^l(\cdot, t_i) : [0, t_i + \bar{\tau}] \longrightarrow [0, 1]$.

Finally, as I show in the benchmark analysis below, the following assumptions guarantee enough heterogeneity in the market to make the analysis interesting. First, λ^h is such that:

$$\frac{\lambda^h}{1-e^{-\lambda^h \eta\kappa}} > \frac{g-r}{\beta} > \frac{\lambda^h}{1-e^{-\lambda^h \bar{\tau}}}. \tag{i}$$

Under this assumption, no bubble can survive in equilibrium in a world where it is common knowledge that there are only h-types. Second, λ^l is such that:

$$\frac{\lambda^l}{1-e^{-\lambda^l \eta\kappa}} < \frac{g-r}{\beta}. \tag{ii}$$

Finally, in the optimistic market, ρ is such that:

$$\rho \in \left(\frac{\eta\kappa - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)}{\bar{\tau} - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)}, \frac{\eta\kappa - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)}{\frac{1}{\lambda^h}\ln\left(\frac{g-r}{g-r-\lambda^h\beta}\right) - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)} \right). \tag{iii}$$

This last assumption, together with assumption ii), guarantees a bubble would burst for exogenous reasons in a world where there are both optimists and pessimists, and this is common knowledge. Assumptions i) and ii) also guarantee that agents do not want to sell out before they become aware of the mispricing.
To summarize, the timing of the game is the following: first, the financial company chooses the signals precision, and then agents, once received the public signal, choose when to sell the asset.

3 Equilibrium Analysis

According to the timing of the game described in the previous section, this is a two-stages game. In stage one, the financial company chooses the signals precision, and then, in stage two, agents, once received the signal and seen the signal precision chosen, decide when to leave the market. Thus, it is convenient for the equilibrium analysis to look first at the second stage of the game, and then at the entire game, where the financial company moves first, before agents play. The structure of this section is the following. In part 1, I will conduct the equilibrium analysis of the second stage of the game, looking at what agents would do for all the possible signals. In part 2, I will conduct the equilibrium analysis of the two-stages game, where the financial company optimally chooses the signals precision before agents make their move.

3.1 Equilibrium Analysis of the Second Stage of the Game

In this part, I refer to the equilibrium concept of t_i*-symmetric perfect Bayesian equilibrium*. This is a perfect Bayesian equilibrium where those agents, who were of the same type, h or l, before t_0, wait the same number of periods after receiving their private information at t_i, before leaving the market.
Let me first introduce some useful notation for computing the probabilities that agents assign to the burst of a bubble.
For every $t \geq t_0$, the aggregate selling pressure of all h-types is given by:

$$s^h(t, t_0) = \int_{t_0}^{\min\{t, t_0+\eta\}} \sigma^h(t, t_i)\, dt_i$$

while the aggregate selling pressure of all l-types is given by:

$$s^l(t, t_0) = \int_{t_0}^{\min\{t, t_0+\eta\}} \sigma^l(t, t_i)\, dt_i.$$

Thus the aggregate selling pressure of all agents at time $t \geq t_0$ is given by:

$$s^P(t, t_0) = s^h(t, t_0)$$

when there are only pessimists, and by:

$$s^O(t, t_0) = \rho s^h(t, t_0) + (1-\rho) s^l(t, t_0)$$

otherwise.

For every given t_0, let

$$T_P^*(t_0) := \inf\left\{t : s^h(t, t_0) \geq k \text{ or } t = t_0 + \bar{\tau}\right\}$$

and

$$T_O^*(t_0) := \inf \left\{ t : \rho s^h(t, t_0) + (1-\rho) s^l(t, t_0) \geq k \text{ or } t = t_0 + \bar{\tau} \right\}$$

be the bursting time of the bubble when there are only pessimists and when there is also a positive fraction of optimists, respectively.

Then, the probability that the bubble bursts before date t, for an h-type, who becomes aware of the mispricing at date t_i, and who knows that there are only pessimists in the market, is:

$$\Pi_P^h(t \mid t_i) := \int_{T_P^*(t_0) < t} d\Phi^h(t_0 \mid t_i).$$

On the other hand, the probabilities that the bubble bursts before date t, for an h-type and for an l-type, who become aware of the mispricing at time t_i, and who know that there is a positive fraction of optimists in the market, are:

$$\Pi_O^h(t \mid t_i) := \int_{T_O^*(t_0) < t} d\Phi^h(t_0 \mid t_i), \; \Pi_O^l(t \mid t_i) = \int_{T_O^*(t_0) < t} d\Phi^l(t_0 \mid t_i)$$

Thus, when there is uncertainty of the market composition, the belief of type (h, t_i) on the bursting time of the bubble is:

$$\Pi^h(t \mid t_i) = \alpha_1 \int_{T_P^*(t_0) < t} d\Phi^h(t_0 \mid t_i) + (1 - \alpha_1) \int_{T_O^*(t_0) < t} d\Phi^h(t_0 \mid t_i)$$

Instead, for type (l, t_i), the belief on the bursting time of the bubble is:

$$\Pi^l(t \mid t_i) = \Pi_O^l(t \mid t_i) = \int_{T_O^*(t_0) < t} d\Phi^l(t_0 \mid t_i).$$

Before conducting the equilibrium analysis of the second stage of the game, I look at the case where, differently from the second stage of the game presented here, there is common knowledge of the market composition. This will serve as a useful benchmark for the equilibrium analysis of the second stage of the game, where there is uncertainty over the market composition. Readers who wish to omit this part may move directly to subsection 3.1.2.

3.1.1 Benchmark Analysis: Common Knowledge of the Market Composition

Proposition 1 shows that, in case the market is populated only by pessimistic agents, and this is common knowledge, no bubble can be sustainable in equilibrium. While AB 2003 rules out this case by assumption, it is important to consider it in order to compare it with the equilibrium analysis when there is uncertainty of the market composition. Proposition 2, instead, considers the novel case where there are different types of agents in the market. It shows that, when the market is populated both by pessimists and optimists, and this is common knowledge, a long bubble is sustainable in equilibrium. In this case, the agents' aggregate selling pressure never reaches the threshold for making the bubble burst, and, thus, the burst happens for exogenous reasons, at time $t_0 + \bar{\tau}$.

Proposition 1 *When the world is populated only by pessimists, and this is common knowledge, there is a unique t_i-symmetric perfect Bayesian equilibrium where agents exit the market as soon as they become aware of the mispricing. No bubble can be sustainable in equilibrium.*

Proof. Type (h, t_i) will choose a selling time $t = t_i + \tau^h$ in order to maximize his expected utility:

$$\int_{t_i}^{t} e^{-rs}(1-\beta)p(s)\pi_P^h(s \mid t_i)ds + e^{-rt}p(t)(1-\Pi_P^h(t \mid t_i)).$$

Taking the first order condition and setting it equal to zero leads to:

$$\frac{\lambda^h}{1-e^{-\lambda^h(\xi-\tau^h)}} = \frac{g-r}{\beta},$$

where ξ is the number of periods after t_0 a bubble lasts before bursting. Notice that, since the first order condition depends on the number of periods after an agent becomes aware of the mispricing, τ^h, but not on a specific t_i, it is the same for every t_i. If agents believe the bubble bursts for endogenous reasons and all other agents wait the same number of periods after becoming aware of the mispricing before exiting the market, i.e. $\xi = \eta k + \tau^h$, then the FOC becomes:

$$\frac{\lambda^h}{1 - e^{-\lambda^h \eta k}} = \frac{g - r}{\beta}.$$

By assumption i), for which $\frac{\lambda^h}{1-e^{-\lambda^h \eta \kappa}} > \frac{g-r}{\beta}$, the LHS is always higher than the RHS, and, thus, agents want to exit as soon as they become aware of the mispricing: $\tau^h = 0$. Since in equilibrium agents must have correct beliefs, the bubble must effectively burst for endogenous reasons. For this to happen, it must hold that:

$$t_0 + \eta k < t_0 + \bar{\tau}.$$

This is always true, as, by assumption, $\eta k < \bar{\tau}$. Since the price and the fundamental value coincide at $t_0 + \eta k$, in this symmetric equilibrium the mispricing never becomes a bubble. The proof of the uniqueness of the equilibrium is left to the Appendix. ■

The reason for the equilibrium described in Proposition 1 is that, in this case, pessimistic agents assign such a high probability to be among the last ones who become aware of the mispricing, that they find it optimal to exit as soon as they receive their private information. The key intuition is that, for them, the risk of staying invested one extra period when they come to know that there is a mispricing is too high, relative to the expected gains from riding it. Next, I introduce the case where the market is populated also by a positive fraction of optimists, and this is common knowledge.

Proposition 2 *When the world is populated both by pessimists and optimists, and this is common knowledge, in the unique t_i-symmetric perfect Bayesian equilibrium all agents stay for a while in the market before exiting. In particular, pessimistic agents stay invested $\tau^h = \bar{\tau} - \frac{1}{\lambda^h} \ln\left(\frac{g-r}{g-r-\lambda^h \beta}\right)$ periods after becoming aware of the mispricing, while optimistic agents stay invested $\tau^l = \bar{\tau} - \frac{1}{\lambda^l} \ln\left(\frac{g-r}{g-r-\lambda^l \beta}\right)$ periods after they become aware of the mispricing. The bubble, once started, always bursts for exogenous reasons.*

Proof. The expected payoff to be maximized for an agent (h, t_i) is:

$$\int_{t_i}^{t} e^{-rs}(1-\beta)p(s)\pi_P^h(s \mid t_i)ds + e^{-rt}p(t)\Pi_P^h(t \mid t_i)$$

and for an agent (l, t_i) is:

$$\int_{t_i}^{t} e^{-rs}(1-\beta)p(s)\pi_O^l(s \mid t_i)ds + e^{-rt}p(t)\Pi_O^l(t \mid t_i).$$

Suppose (h, t_i) and (l, t_i) believe that the bubble bursts at time $t_0 + \xi$. Then, the first order conditions for types (h, t_i) and (l, t_i) are:

$$\frac{\lambda^h}{1-e^{-\lambda^h(\xi-\tau^h)}} = \frac{g-r}{\beta} \qquad \text{and} \qquad \frac{\lambda^l}{1-e^{-\lambda^l(\xi-\tau^l)}} = \frac{g-r}{\beta},$$

respectively. If agents believe the bubble bursts for exogenous reasons, then $\xi = \bar{\tau}$ and the first order conditions become:

$$\frac{\lambda^h}{1-e^{-\lambda^h(\bar{\tau}-\tau^h)}} = \frac{g-r}{\bar{\beta}} \qquad \text{and} \qquad \frac{\lambda^l}{1-e^{-\lambda^l(\bar{\tau}-\tau^l)}} = \frac{g-r}{\bar{\beta}}.$$

Since, by assumption, $\frac{g-r}{\beta} > \frac{\lambda^h}{1-e^{-\lambda^h\bar{\tau}}}$, for an h-type the LHS starts lower than the RHS and it is increasing in τ^h. Similarly, since by assumption $\frac{g-r}{\beta} > \frac{\lambda^l}{1-e^{-\lambda^l\eta\kappa}}$, also for an l-type the LHS starts lower than the RHS and it is increasing in τ^l. Thus, the best response for type (h, t_i) is:

$$\tau^h = \bar{\tau} - \frac{1}{\lambda^h}\ln\left(\frac{g-r}{g-r-\lambda^h\beta}\right)$$

and the best response for type (l, t_i) is:

$$\tau^l = \bar{\tau} - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right).$$

For agents to hold correct beliefs, the bubble must effectively burst for exogenous reasons, that is:

$$t_0 + \eta\kappa + \rho\tau^h + (1-\rho)\tau^l \geq t_0 + \bar{\tau}.$$

Substituting the optimal τ^h and τ^l, it must be true that:

$$\eta\kappa \geq \rho\left[\frac{1}{\lambda^h}\ln\left(\frac{g-r}{g-r-\lambda^h\beta}\right)\right] + (1-\rho)\left[\frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)\right].$$

This is always true, since, by assumption, $\rho < \frac{\eta k - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)}{\frac{1}{\lambda^h}\ln\left(\frac{g-r}{g-r-\lambda^h\beta}\right) - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)}$.
Thus, τ^h and τ^l define a symmetric equilibrium where the bubble bursts for exogenous reasons. The proof of the uniqueness of the symmetric equilibrium is left to the Appendix. ■

The equilibrium in Proposition 2 has two interesting features. First, optimists stay longer invested in the market, once they become aware of the mispricing, compared to pessimists. The key intuition is that, for optimists, in every period, even after they become aware of the mispricing, the risk from staying invested one extra period is lower than the one perceived by pessimists. Consequently, they find it optimal to ride the bubble for longer. Second, pessimistic agents, differently from the world without optimists, find it optimal to ride the bubble for a while after having become aware of the mispricing. They key idea is that, the existence of optimists, who stay invested in the market for longer, lowers the risk perceived by pessimists of an immediate burst, and, thus, makes them willing to ride the bubble for a while.

Thus, from the benchmark analysis we can conclude that, with common knowledge of the true state of the world, in equilibrium, no bubble is sustainable in a world with only pessimists, while a long bubble is sustainable in the world with at least a fraction $(1-\rho)$ of optimists. In this case, the bubble would burst for exogenous reasons. In the next subsection, I show how the equilibrium strategies change when there is uncertainty of the market composition, that is the case of the second stage of the game presented in this paper.

3.1.2 Equilibrium Analysis of the Second Stage of the Game

In this part, I look at the equilibrium analysis of the second stage of the game presented in this paper, where there is uncertainty about the true composition of the market, and I show that a bubble can start even when no bubble is sustainable in the equilibrium with common knowledge of the true proportions. In particular, there are two cases. First, if α_1, the updated probability that pessimistic agents assign to a pessimistic world, is lower than a given threshold, then there is a unique equilibrium where the bubble bursts for endogenous reasons if the world is pessimistic, and for exogenous reasons if the world is populated also by optimists. Indeed, pessimistic agents stay longer invested than if they know the world is populated only by pessimists. Thus, in case the world turns out to be populated only by pessimists, their initial beliefs that there could have been some optimists in the market make a bubble possible in equilibrium. In this sense, the bubble is self-fulfilling. Indeed, pessimists' strategies alone make the bubble begin. On the other side, if the world turns out to be populated also by optimists, the bubble lasts until it bursts for exogenous reasons, like in the benchmark case.

Second, if α_1 is higher than the same threshold, then the unique equilibrium features a bubble only if there are some optimistic agents, and the bubble is directly caused by them, as pessimists exit immediately the market. In this case, pessimistic agents strongly believe there are no optimists in the world, and behave as if they were sure of the world being populated only by pessimists.[3]

[3] As shown in the Appendix, when α_1 is exactly equal to the threshold, in addition to the two equilibria described above, the one in which pessimists exit immediately and the other where they stay invested for a while after becoming aware of the mispricing, there is a multiplicity of equilibria where pessimists stay invested for some time in this interval, and optimists behave accordingly. In these equilibria, the bubble always bursts for endogenous reasons, no matter what the true composition of the market is. All the equilibria in this third case can be ranked both in terms of duration of the bubble, and in terms of Pareto efficiency. In particular, the equilibrium where the bubble bursts for endogenous reasons if the world is pessimistic, and for exogenous reasons otherwise, being the longest, ex-ante Pareto dominates all the other equilibria.

Proposition 3 *Let*

$$\overline{\alpha} = \frac{e^{\frac{\lambda^h}{\rho}\left[\eta k - \frac{(1-\rho)ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)}{\lambda^l}\right]} - \frac{g-r}{g-r-\beta\lambda^h}}{e^{\frac{\lambda^h}{\rho}\left[\eta k - \frac{(1-\rho)ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)}{\lambda^l}\right]} - e^{\lambda^h \eta k}}.$$

If $\alpha_1 < \overline{\alpha}$, then there exists a unique t_i-symmetric perfect bayesan equilibrium where the bubble bursts for endogenous reasons if the world is pessimistic, and for exogenous reasons otherwise. Moreover, for every type (h, t_i) the optimal strategy makes him wait:

$$\tau^h = \bar{\tau} - \frac{1}{\lambda^h} ln\left[\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)} - \frac{\alpha_1}{1-\alpha_1} e^{\lambda^h \eta k}\right]$$

periods after having become aware of the mispricing, before selling out. For type (l, t_i) the optimal strategy makes him wait:

$$\tau^l = \bar{\tau} - \frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)$$

periods after t_i to sell out.

Proof . Type (h, t_i) chooses a selling time t in order to maximize his expected utility:

$$\int_{t_i}^{t} e^{-rs}(1-\beta)p(s)\pi^h(s \mid t_i)ds + e^{-rt}p(t)(1 - \Pi^h(t \mid t_i)).$$

For type (l, t_i), instead, the expected utility, he wants to maximize, is:

$$\int_{t_i}^{t'} e^{-rs}(1-\beta)p(s)\pi^l(s \mid t_i) + e^{-rt}p(t)(1 - \Pi^l(t \mid t_i)).$$

Before finding the optimal exit time $t = t_i + \tau^h$ and $t' = t_i + \tau^l$, let me define the hazard rate functions for h-types and for l-types as:

$$h^h(t \mid t_i) := \frac{\pi^h (t \mid t_i)}{1 - \Pi^h (t \mid t_i)} \qquad \text{and} \qquad h^l(t \mid t_i) := \tfrac{\pi^l(t|t_i)}{1-\Pi^l(t|t_i)}.$$

Then, setting the first order conditions equal to zero:

$$h^h(t_i + \tau^h \mid t_i) = \frac{g-r}{\beta} \qquad \text{and} \qquad h^l(t_i + \tau^l \mid t_i) = \tfrac{g-r}{\beta}.$$

The first order conditions tell that the optimal selling time for an agent is when the hazard rate from riding the bubble one extra period is equal to the benefit obtained by selling the stock one period later. The right hand sides of the first order conditions, not depending on t_i, are the same for every agent of the same type, h or l. From the FOC for an h-type, it must hold that:

$$\frac{\lambda^h}{1 - \frac{1}{\alpha_1 e^{\lambda^h(\xi - \tau^h)} + (1-\alpha_1) e^{\lambda^h(\xi' - \tau^h)}}} = \frac{g-r}{\beta},$$

where ξ is the number of periods after t_0 a bubble lasts before bursting in a world populated by pessimists only, and ξ' is the number of periods after t_0 a bubble lasts before bursting in case there are also optimists in the world. If agents believe the bubble bursts for endogenous reasons when the world is pessimistic and for exogenous reasons otherwise, and that other agents use symmetric strategies, then they think that $\xi = \eta k + \tau^h$ and $\xi' = \bar{\tau}$. Substituting these terms in the FOC leads to:

$$\frac{\lambda^h}{1 - \frac{1}{\alpha_1 e^{\lambda^h \eta k} + (1-\alpha_1) e^{\lambda^h(\bar{\tau} - \tau^h)}}} = \frac{g-r}{\beta}.$$

Since this is valid for every t_i, every h-type waits the same number of periods τ^h after having become aware of the mispricing to sell the asset. By the assumption on $\alpha_1 < \bar{\alpha}$, the LHS of the above equation starts below the RHS and is strictly

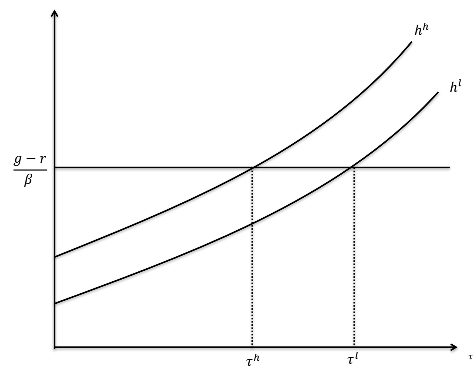

Figure 1: Optimal exit times for pessimists, τ^h, and optimists, τ^l, when $\alpha_1 < \bar{\alpha}$. In the graph, h^h represents the hazard rate for a pessimistic agent, while h^l represents the hazard rate for an optimist.

increasing in τ^h. Thus, the unique τ^h that satisfies this equation is:

$$\tau^h = \bar{\tau} - \frac{1}{\lambda^h} ln \left[\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h \beta)} - \frac{\alpha_1}{1-\alpha_1} e^{\lambda^h \eta k} \right].$$

For type (l, t_i) from the FOC, and since $\xi' = \bar{\tau}$:

$$\frac{\lambda^l}{1-e^{-\lambda^l(\bar{\tau}-\tau^l)}} = \frac{g-r}{\beta}.$$

Thus, every l-type waits the same number of periods τ^l after having become aware of the mispricing before selling the asset:

$$\tau^l = \bar{\tau} - \frac{1}{\lambda^l} ln \left(\frac{g-r}{g-r-\lambda^l \beta} \right).$$

Lastly, these strategies must lead to an equilibrium where the bubble bursts for endogenous reasons, if the world is populated only by pessimists, and for exogenous reasons otherwise, that is:

$$t_0 + \eta\kappa + \tau^h < t_0 + \bar{\tau}$$
$$t_0 + \eta\kappa + \rho\tau^h + (1-\rho)\tau^l \geq t_0 + \bar{\tau}.$$

Substituting the optimal τ^h and τ^l in the above conditions, it leads to:

$$e^{\lambda^h \eta k} < \frac{g-r}{g-r-\beta\lambda^h}$$

and

$$\eta k - \frac{(1-\rho)}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right) \geq \frac{\rho}{\lambda^h} ln\left[\frac{g-r}{(1-\alpha_1)(g-r-\beta\lambda^h)} - \frac{\alpha_1}{1-\alpha_1}e^{\lambda^h \eta k}\right].$$

The first inequality is true by assumption i). If $\alpha_1 < \overline{\alpha}$, also the second inequality is always satisfied. The proof of the uniqueness of the symmetric equilibrium is left to the Appendix. ■

The equilibrium in Proposition 3 has some interesting features. First, like in the benchmark case with common knowledge of agents' beliefs, optimistic agents stay longer invested after becoming aware of a mispricing, compared to pessimistic agents. The reason is that, according to the first order conditions, different types exit the market when their hazard rates reach the same level. As figure 1 shows, the hazard rate for a pessimist is always above the one for an optimist. This means that a pessimist, who received the information on the mispricing at the same time of an optimist, stays shorter invested in the market. The second interesting feature of this equilibrium is that, differently from the benchmark case, where a bubble would not be sustainable, uncertainty about the market composition makes a bubble possible also in a world populated only by pessimists. The key intuition is that the pessimists' belief that there could be some optimists in the market, who would sustain a bubble for a while, is enough to make pessimists more optimistic about the length of a bubble. Thus, the pessimists' belief that there are some optimists in the market lowers their perceived risk of an immediate burst, making them willing to ride the bubble for a while. Moreover, since the optimal strategy for a pessimist is increasing in $(1-\alpha_1)$, the higher the probability a pessimist assigns to the world being optimistic, the longer he stay invested in the market after becoming aware of the mispricing. Indeed, when $\alpha_1 = 0$, pessimists stay invested the same number of periods as in the benchmark case in Proposition 2. Finally, optimists know what the true composition of the market is, and they correctly believe that pessimists stay invested long enough to sustain a bubble that bursts for exogenous reasons in case the market is optimistic. For this reason, they choose the same strategy they would pick in the benchmark case. Next, I introduce the other case, when pessimists are too pessimistic about the market composition, i.e., when $\alpha_1 > \overline{\alpha}$.

Proposition 4 *Consider $\overline{\alpha}$ from Proposition 3. If $\alpha_1 > \overline{\alpha}$, then there exists a unique t_i-symmetric perfect Bayesian equilibrium where the bubble bursts immediately if the world is populated only by pessimistic agents, and it bursts for endogenous reasons otherwise. Every type (h, t_i) exits as soon as he his informed of the mispricing. For type (l, t_i) the optimal strategy is to wait:*

$$\tau^l = \frac{\eta k}{\rho} - \frac{1}{\rho\lambda^l} ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)$$

periods after t_i to sell out.

Proof . Type (h, t_i) will choose a selling time t such that the expected utility:

$$\int_{t_i}^{t} e^{-rs}(1-\beta)p(s)\pi^h(s \mid t_i)ds + e^{-rt}p(t)(1-\Pi^h(t \mid t_i))$$

is maximized. For type (l, t_i) the expected utility to be maximized is:

$$\int_{t_i}^{t'} e^{-rs}(1-\beta)p(s)\pi^l(s \mid t_i) + e^{-rt}p(t)(1-\Pi^l(t \mid t_i)).$$

To find the optimal exit time $t = t_i + \tau^h$ and $t' = t_i + \tau^l$, and thus the optimal τ^h and τ^l for the two types, it is necessary to set the first order conditions equal to zero. For an h-type, this leads to:

$$\frac{\lambda^h}{1 - \frac{1}{\alpha_1 e^{\lambda^h(\xi-\tau^h)} + (1-\alpha_1)e^{\lambda^h(\xi'-\tau^h)}}} = \frac{g-r}{\beta}.$$

If agents believe the bubble always bursts for endogenous reasons, and that other agents use symmetric strategies, then $\xi = \eta k + \tau^h$ and $\xi' = \eta k + \rho\tau^h + (1-\rho)\tau^l$. Substituting these terms and also the correct belief about τ^l, $\tau^l = \frac{\eta k}{\rho} - \frac{1}{\rho\lambda^l} ln(\frac{g-r}{g-r-\lambda^l\beta})$, in the FOC leads to:

$$\frac{\lambda^h}{1 - \frac{1}{\alpha_1 e^{\lambda^h \eta k} + (1-\alpha_1) e^{\lambda^h \left[\eta k - (1-\rho)\tau^h + (1-\rho)\frac{\eta k}{\rho} - (1-\rho)\frac{1}{\rho\lambda^l} ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)\right]}}} = \frac{g-r}{\beta}.$$

Since this is valid for every t_i, every h-type waits the same number of periods τ^h after having become aware of the mispricing. If $\alpha_1 > \overline{\alpha}$, then the LHS starts above the RHS and it is strictly increasing in τ^h. This means that pessimistic agents find it optimal to exit immediately the market, that is $\tau^h = 0$.
For type (l, t_i) from the FOC and since $\xi' = \eta k + \rho\tau^h + (1-\rho)\tau^l$ and $\tau^h = 0$:

$$\frac{\lambda^l}{1 - e^{-\lambda^l(\eta k - \rho\tau^l)}} = \frac{g-r}{\beta}.$$

Thus, every l-type waits the same number of periods:

$$\tau^l = \frac{\eta k}{\rho} - \frac{1}{\rho\lambda^l} ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)$$

after having become aware of the mispricing before selling the asset.

These strategies must lead to an equilibrium where the bubble bursts for endogenous reasons in both worlds:

$$t_0 + \eta\kappa + \tau^h < t_0 + \bar{\tau}$$
$$t_0 + \eta\kappa + \rho\tau^h + (1-\rho)\tau^l < t_0 + \bar{\tau}$$

Substituting the optimal τ^h and τ^l in the above conditions leads to:

$$\eta k < \bar{\tau},$$

$$\eta k - (1-\rho)\frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right) < \rho\bar{\tau}.$$

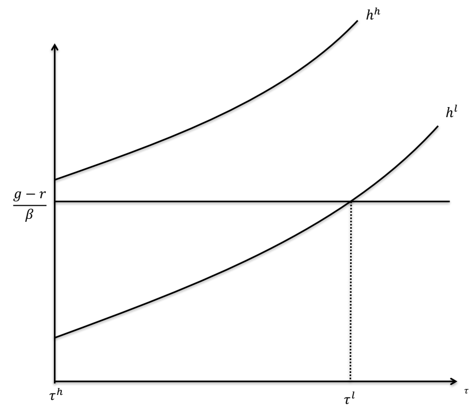

Figure 2: Optimal exit times for pessimists, τ^h, and optimists, τ^l, when $\alpha_1 > \bar{\alpha}$. In the graph, h^h represents the hazard rate for a pessimistic agent, while h^l represents the hazard rate for an optimist.

The first inequality is always satisfied. The second is true under assumption iii) on $\rho > \frac{\eta\kappa - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)}{\bar{\tau} - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)}$.
Thus, in this symmetric equilibrium the bubble does not even start if the world is pessimistic and it bursts for endogenous reasons otherwise. The proof of the uniqueness of the equilibrium is left to the Appendix. ■

Let me discuss some interesting features of the equilibrium described in Proposition 4. First, also in this equilibrium optimistic agents exit later than pessimists. Nevertheless, pessimists choose to leave immediately the market because they assign a very low probability to the existence of optimists in the market. Consequently, as shown in Figure 2, the risk of an immediate burst perceived by the pessimistic agents is still too high compared to the potential benefits from riding the bubble for a while. This makes them willing to exit the market as soon as they become aware of the mispricing. Second, differently from the previous case described in Proposition 3, now optimists' strategies are influenced by the pessimists' behavior. Indeed, optimistic agents are not willing to stay invested in the market as long as before, because they correctly predict that the pessimists' exit strategies contribute to an early burst of the bubble.

These results contribute to our understanding of the formation and duration of financial bubble. Moreover, Proposition 3 and 4 can shed light on why agents react to some, but not to another, "non-fundamental" news, i.e., news that does not contain any information on the fundamental value of an asset. Suppose that there is a bubble in act and that the market is actually made of both pessimists and optimists. Suppose, moreover, that the market is reached by public unexpected negative news that reveals something about the market sentiment. Agents use such news as a signal on the beliefs of other agents and, consequently, using Bayes' rule, they update their belief about the true proportion of pessimists in the market. If, before the news, the belief α_1 is below the threshold $\bar{\alpha}$, and, after the news, the updated belief is above the threshold, then, according to the propositions above, pessimistic agents change their strategy and immediately exit the market. The change in their strategy causes the market to move from the equilibrium with the bubble bursting for exogenous reasons to the one with the bubble bursting for endogenous reasons, and the bubble may deflate. On the other hand, if beliefs about the prevailing market sentiment were already above the threshold, the same news would not change the equilibrium outcome and the bubble would not deflate. Suppose instead the news is positive. If agents' prior beliefs were above the threshold, while the updated ones are below, the news would make pessimistic agents stay longer invested and the equilibrium would change from the one with the bubble bursting for endogenous reasons to the one with the bubble bursting for exogenous reasons. The same news would leave the equilibrium outcome unchanged if the initial beliefs were already below the threshold. Thus, the model in this paper can help to explain why the same news has different consequences on the market outcome depending on agents' initial beliefs about the market sentiment.

3.2 Equilibrium Analysis of the Two-Stages Game

In this section, I conduct the equilibrium analysis of the two-stages game, where, first, the financial company chooses the signal precision, and then agents choose when to sell the asset. I look at the *agents' most preferred responsive equilibria*. These are t_i-symmetric perfect Bayesian equilibria, where all agents react to the signals sent by the company and, once received the signal, they always settle on the

Pareto dominant outcome of the second stage of the game. In the following, I show that in equilibrium the company finds it optimal to manipulate agents' second order beliefs, inducing them to stay on average longer invested in the market. Indeed, as I have shown in the previous section, a change in the belief α_0 influences agents' strategies. This means that, any information influencing α_0 influences also agents' strategies. Consequently, influencing α_0 in its preferred direction, the company increases the probability of a new financial bubble, or a longer one, in equilibrium. In particular, proposition 5 shows that, when α_0 is higher than a certain value α^*, in the unique agents' most preferred responsive equilibrium of the game, the company wants to convey some, but not all, information to agents to manipulate their beliefs and to induce them to stay longer invested in the market. In this case, for the company it is optimal to aggregate data in a way that if the world is optimistic, $(\omega = O)$, the signal is always O, while if the world is pessimistic, the data aggregation is less precise, and the signal is P with probability $1 - \frac{(1-\alpha_0)\alpha^*}{(1-\alpha^*)\alpha_0} < 1$ only. With this signals structure, the company is able to influence agents' strategies and, consequently, to increase the probability of having a bubble in the market, or to strengthen an existing one. Instead, if $\alpha_0 \leq \alpha^*$, there is nothing the company wants to do in terms of persuasion, as its expected utility cannot be increased any further. Indeed, in this case in equilibrium agents are already sustaining the most profitable bubble for the financial company in every state of the world. Thus, in this case, there is no need for the manipulator to intervene.

Proposition 5 *Let α^* solve $u(\alpha) = -u'(\alpha)\alpha(1-\alpha)$. If $\alpha_0 > \alpha^*$, then, in the unique agents' most preferred responsive equilibrium, the company chooses the following public signal structure:*

$$pr(S = P \mid \omega = P) = 1 - \frac{(1-\alpha_0)\alpha^*}{(1-\alpha^*)\alpha_0}$$

and

$$pr(S = P \mid \omega = O) = 0.$$

If agents receive the P signal, pessimists exit immediately and optimists stay:

$$\tau^l = \frac{\eta k}{\rho} - \frac{1}{\rho\lambda^l} ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)$$

periods after t_i, before selling out. The bubble bursts for endogenous reasons if the market is made of both pessimists and optimists, and does not even start if the market is pessimistic. If agents receive the O signal, instead, pessimists stay:

$$\tau^h = \bar{\tau} - \frac{1}{\lambda^h} ln\left[\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)} - \frac{\alpha_1}{1-\alpha_1}e^{\lambda^h \eta k}\right]$$

periods after having become aware of the mispricing before selling out, while optimists wait:

$$\tau^l = \bar{\tau} - \frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)$$

periods after t_i before selling out. The bubble bursts for endogenous reasons if the world is pessimistic ($\rho = 1$), and for exogenous reasons otherwise.
If $\alpha_0 \leq \alpha^*$, the company does not want to convey any information.

Proof. I use backward induction to find the agents' most preferred responsive equilibrium. From proposition 3 and 4, it is possible to derive what agents do for every generic belief α. If $\alpha > \overline{\alpha}$, then pessimistic agents exit immediately the market and a bubble cannot even start in case the market is composed by pessimists only. If $\alpha < \overline{\alpha}$, instead, a bubble is always sustainable in equilibrium, as agents stay for a while invested even if they know there is a bubble. Finally, when $\alpha = \bar{\alpha}$, as shown in Lemma 1 in the Appendix, it is Pareto dominant to behave in the same way as when $\alpha < \overline{\alpha}$. Then, I focus on the best action for a company, that moves first. Indirectly, in order to maximize its expected utility, the choice set of the financial company is A_1, the set of all posterior beliefs over the possible states of the world. For this reason, it is useful to derive the utility function of the company, and its expected utility, as a function of a generic belief α. When $\omega = P$, for every $\alpha > \overline{\alpha}$, $u(\alpha) = 0$, as in this case a bubble does not even start in equilibrium. On the other side, when $\omega = P$, for every $\alpha \leq \overline{\alpha}$, $u(\cdot)$ is positive, concave and

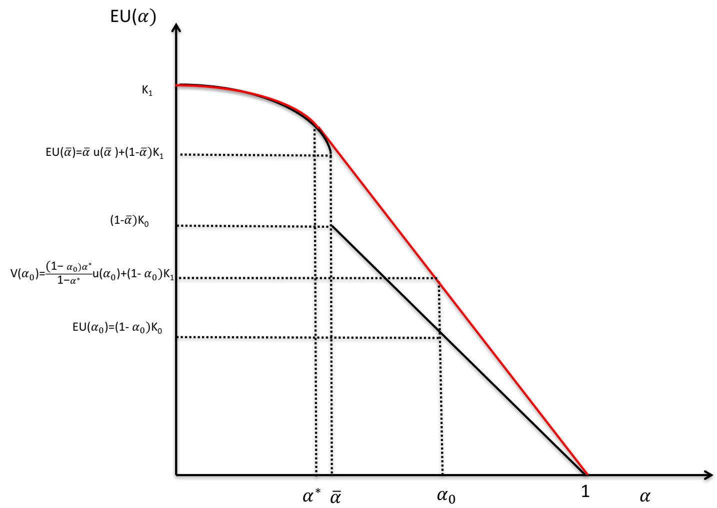

Figure 3: $\alpha^* < \overline{\alpha}$. The black line is the expected utility function of the company as a function of beliefs about the market sentiment. The red line, instead, represents the convex hull of the company expected utility function. When $\alpha_0 > \alpha^*$, then $V(\alpha_0) > EU(\alpha_0)$ and the company finds it profitable to manipulate the market.

decreasing in α, as by assumption $u(\cdot)$ is concave or linear in the length of the bubble τ and in equilibrium τ is positive, decreasing and concave in α. If $\omega = O$, then for every $\alpha > \overline{\alpha}$, $u(\alpha) = K_0$, where $K_0 > 0$ is a constant, as the length of the bubble does not depend on α. Finally, if $\alpha \leq \overline{\alpha}$, then $u(\alpha) = K_1$, where $K_1 > K_0$ is a constant, as in this case the length of the bubble is always $\bar{\tau} - \eta k$. Then, the expected utility of the company, in terms of α, is $EU(\alpha) = (1-\alpha)K_0$, if $\alpha > \overline{\alpha}$, and $EU(\alpha) = \alpha u(\alpha) + (1-\alpha)K_1$ otherwise. By the assumptions made on the utility function $u(\cdot)$, the expected utility function is nonnegative, has a step discontinuity at $\alpha = \alpha_0$, is decreasing in α, and is concave for $\alpha \leq \alpha_0$ and linear for $\alpha > \alpha_0$. When the company sends a signal, by corollary 2 in KG, the optimal signal is the one that induces an expected utility equal to $V(\alpha_0)$, the value of the convex hull of the expected utility, computed at the prior belief. Indeed, this is the signal structure that maximizes the expected utility of the company, subject to Bayesian plausibility. To find the convex hull of the expected utility, I need to find the unique value α^* for which the expected utility is tangent to the line starting at $(\alpha = 1, EU(\alpha = 1) = 0)$, with slope $\frac{-\alpha u(\alpha)-(1-\alpha)K_1}{1-\alpha}$. In other words, α^* must solve the equation $\alpha u'(\alpha) + u(\alpha) - K_1 = \frac{-\alpha u(\alpha)-(1-\alpha)K_1}{1-\alpha}$. Thus, α^* is such that $\frac{u(\alpha^*)}{u'(\alpha^*)} = -\alpha^*(1-\alpha^*)$, where, by assumption, $u'(\alpha^*) < 0$. Let me first show that $\alpha^* > 0$. The slope of the expected utility at $\alpha = 0$ is $u(\alpha = 0) - K_1$. This is always greater than $-K_1$, the slope of the line starting at $(\alpha = 1, EU(\alpha = 1) = 0)$, and passing through the point $(\alpha = 0, EU(\alpha = 0) = K_1)$. Since the expected utility is concave, this means that the tangency point between the two curves must happen for an $\alpha^* > 0$. Thus, if $\alpha^* \geq \overline{\alpha}$, the convex hull of the expected utility is given by the following function: $V(\alpha) = \alpha u(\alpha) + (1-\alpha)K_1$, if $\alpha \leq \overline{\alpha}$, and $V(\alpha) = \frac{(1-\alpha)\overline{\alpha}}{1-\overline{\alpha}} u(\overline{\alpha}) + (1-\alpha)K_1$, if $\alpha > \overline{\alpha}$. If $\alpha^* < \overline{\alpha}$, then the convex hull of the expected utility is given by the following function: $V(\alpha) = \alpha u(\alpha) + (1-\alpha)K_1$, if $\alpha \leq \alpha^*$, and $V(\alpha) = \frac{(1-\alpha)\alpha^*}{1-\alpha^*} u(\alpha^*) + (1-\alpha)K_1$, if $\alpha > \alpha^*$. Let me focus on the case, represented in figure 3, where $\alpha^* < \overline{\alpha}$. A similar result would hold for $\alpha^* \geq \overline{\alpha}$. The next step is to compute the convex hull at the prior belief α_0. When prior beliefs are pessimistic enough, i.e. $\alpha_0 > \alpha^*$, the convex hull computed at the prior is $V(\alpha_0) = \frac{(1-\alpha_0)\alpha^*}{1-\alpha^*} u(\alpha^*) + (1-\alpha_0)K_1$. Then, since $\alpha_1(\omega = P \mid S = P) = 1$, it must be $pr(S = O) = \frac{1-\alpha_0}{1-\alpha^*}$ and, also, $pr(S = P \mid \omega = P) = 1 - \frac{(1-\alpha_0)\alpha^*}{(1-\alpha^*)\alpha_0}$. From $pr(S = O) = \frac{1-\alpha_0}{1-\alpha^*}$, it is possible to derive $pr(S = O \mid \omega = O) = 1$ and, consequently, $pr(S = P \mid \omega = O) = 0$. Finally, I compute the posterior beliefs and check they satisfy Bayesian plausibility. Using Bayes rule, the posterior beliefs are $\alpha_1(\omega = P \mid s = P)) = 1, \alpha_1(\omega = O \mid s = P)) = 0, \alpha_1(\omega = O \mid s = 0)) = 1-\alpha^*$, and $\alpha_1(\omega = P \mid s = 0)) = \alpha^*$. These posterior beliefs are Bayesian plausible, as Bayesian plausibility requires $\alpha_1(\omega = P \mid s = P)pr(s = P) + \alpha_1(\omega = P \mid s = O)pr(s = O) = \alpha_0$. Moreover, $V(\alpha_0)$ is greater than the expected utility when no signals are sent, as $\frac{(1-\alpha_0)\alpha^*}{1-\alpha^*} u(\alpha^*) + (1-\alpha_0)K_1 > (1-\alpha_0)K_0$. Thus, if $\alpha_0 > \alpha^*$, the optimal signal structure chosen by the company is $pr(S = P \mid \omega = P) = 1 - \frac{(1-\alpha_0)\alpha^*}{(1-\alpha^*)\alpha_0}$ and $pr(S = P \mid \omega = O) = 0$. According to the equilibrium analysis in Section 2, with these beliefs, if they receive an O signal, pessimistic agents stay $\tau^h = \bar{\tau} - \frac{1}{\lambda^h} ln\left[\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)} - \frac{\alpha_1}{1-\alpha_1} e^{\lambda^h \eta k}\right]$ periods after having become aware of the mispricing before selling out, while optimists wait $\tau^l = \bar{\tau} - \frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)$ periods after t_i before selling out. The bubble bursts for endogenous reasons if the world is pessimistic, and for exogenous reasons otherwise. If they receive a P signal, pessimistic agents exit immediately, and optimistic agents stay $\tau^l = \frac{\eta k}{\rho} - \frac{1}{\rho\lambda^l} ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)$ periods after t_i, before selling out. The bubble bursts for endogenous reasons if the market is made of both pessimists and optimists, and does not even start if the market is pessimistic. On the other side, as it is clear from figure 3, if $\alpha_0 \leq \alpha^*$, the manipulator cannot improve his expected payoff inducing in pessimistic agents a posterior belief different from the prior. In this case he finds it optimal to choose uninformative signals. Thus, he chooses the following uninformative public signal structure: $pr(S = P \mid \omega = P) = 0.5$ and $pr(S = P \mid \omega = O) = 0.5$. By construction, the equilibrium is unique. ■

From Proposition 5 it is clear that, when agents are pessimistic enough about the market sentiment ($\alpha_0 > \alpha^*$), a financial information company has interest in manipulating these second order beliefs. The key reason is that partially revealing information on the market composition to these agents makes them more optimistic, on average, in their beliefs about other agents. The signals precision described above is such that, if agents receive a P signal, they will update to 1 their belief that the world is populated only by pessimists. On the other hand, if they receive an O signal, they update to $1 - \alpha^* < 1$ their belief on the world being populated also by optimistic agents. This way, if the market is populated only by pessimists, when agents receive an O signal, they stay longer invested in the market and their strategy gives rise to a bubble, that would not arise without manipulation. If the market is populated also by optimists, an O signal will induce pessimists to stay even longer invested, making a bubble last much longer than without manipulation, until a burst for exogenous reasons. On the other side, if agents receive a P signal, they behave as in the case with no signals. Thus, sending a signal improves, on average, the company expected utility, because it increases the probability of having a new bubble in the market, or strengthening an existing one. In addition, any other signal precision would be less profitable for the financial company. Intuitively, an higher precision of the P signal, when the state of the world is P, would decrease the total probability of an O signal, and this is less profitable for the company. At the same time, a lower precision would make $\alpha_1((\omega = P \mid s = O))) > \alpha^*$, not persuading pessimistic agents to stay long enough invested in the market when they see the O

signal, resulting in lower expected profits for the company.
In conclusion, the result in this section shows that, with second order uncertainty, there is room for a new type of market manipulation. Traditional models show that market manipulation works by misleading agents' beliefs about the fundamentals of a stock. Instead, when there are heterogeneity in beliefs and second order uncertainty, there is another channel through which manipulation could work: by misleading agents' beliefs about others' beliefs on the fundamentals.

4 Conclusions

This paper argues that, during periods of financial turbulence, rational agents are uncertain not only about the fundamental value of a stock, but also about the prevailing market sentiment, i.e., other agents' beliefs about the fundamental value of an asset. This double uncertainty, being reflected in prices, can lead to self-fulfilling bubbles, that is, bubbles based only on the belief that other, more optimistic, agents are sustaining the price above the fundamentals. Moreover, unexpected public news about the prevailing market sentiment can influence agents' strategies, leading to new financial bubble or to the deflation of a current one. Finally, and most importantly, with second order uncertainty about fundamentals, that is uncertainty about other agents' beliefs about the fundamental value of an asset, market manipulation can work through a novel channel, the manipulation of second order beliefs, and lead to longer periods of mispricing. Thus, market manipulation could work in an environment where there are differences in opinions caused by differences in prior beliefs, even if noone has insider information about fundamentals.

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Appendix

Proof (Uniqueness of the symmetric equilibrium in Proposition 1).

If the symmetric equilibrium described in Proposition 1 is unique, then there cannot be another symmetric equilibrium, where the bubble bursts for exogenous reasons. If such an equilibrium exists, then agents must believe the bubble bursts for exogenous reasons, i.e $\xi = \bar{\tau}$, and the FOC becomes: $\frac{\lambda^h}{1-e^{-\lambda^h(\bar{\tau}-\tau^h)}} = \frac{g-r}{\beta}$. Then, pessimists would choose to exit $\tau^h = \bar{\tau} - \frac{1}{\lambda^h} ln(\frac{g-r}{g-r-\beta\lambda^h})$ periods after they became aware of the mispricing. Since in equilibrium agents must hold correct beliefs, let us check the bubble effectively bursts for exogenous reasons: $t_0 + \eta k + \bar{\tau} - \frac{1}{\lambda^h} ln\left(\frac{g-r}{g-r-\beta\lambda^h}\right) \geq t_0 + \bar{\tau}$ This is not possible, since, by assumption, $\frac{\lambda^h}{1-e^{-\lambda^h\eta\kappa}} > \frac{g-r}{\beta}$. Thus, there is a unique symmetric equilibrium where agents exit as soon as they become aware of the mispricing and a bubble does not even start. ■

Proof (Uniqueness of the symmetric equilibrium in Proposition 2).

If the symmetric equilibrium described in Proposition 2 is unique, then there cannot be any other equilibrium, symmetric in t_i, where the bubble bursts for endogenous reasons. If such an equilibrium exists, then the FOC for an h-type and for an l-type would become: $\frac{\lambda^h}{1-e^{-\lambda^h[\eta k-(1-\rho)\tau^h+(1-\rho)\tau^l]}} = \frac{g-r}{\beta}$ and $\frac{\lambda^l}{1-e^{-\lambda^l(\eta k+\rho\tau^h-\rho\tau^l)}} = \frac{g-r}{\beta}$, respectively. For both equations, the LHS is strictly increasing in an agent's strategy τ^h and τ^l, respectively. Then, there can be four possible case. The first case is when only pessimistic agents exit immediately the market, with the optimal strategies $\tau^{h*} = 0$ and $\tau^{l*} > 0$. This happens when, for an h-type, the LHS is greater or equal to the RHS at $\tau^h = 0$, that is when: $\frac{\lambda^h}{1-e^{-\lambda^h[\eta k+(1-\rho)\tau^l]}} \geq \frac{g-r}{\beta}$. In case $\tau^{h*} = 0$, then the optimists would find it optimal to stay $\tau^{l*} = \frac{1}{\rho}[\eta k - \frac{1}{\lambda^l} ln(\frac{g-r}{g-r-\beta\lambda^l})]$ periods after they become aware of the mispricing. Substituting the optimal waiting time for optimists in the above inequality for a pessimist, leads to: $\frac{\lambda^h}{1-e^{-\lambda^h\frac{1}{\rho}\left[\eta k-(1-\rho)\frac{1}{\lambda^l}ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right]}} \geq \frac{g-r}{\beta}$. This inequality cannot be satisfied since $\eta k > (1-\rho)\frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right) + \frac{\rho}{\lambda^h} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)$, by the assumption $\rho < \frac{\eta k - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)}{\frac{1}{\lambda^h}\ln\left(\frac{g-r}{g-r-\lambda^h\beta}\right)-\frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)}$. Similarly, by assumption ii) on optimists, for which $\frac{\lambda^l}{1-e^{-\lambda^l\eta k}} < \frac{g-r}{\beta}$, it is not possible that both $\tau^{h*} = 0$ and $\tau^{l*} = 0$, as it would be required at the same time: $\frac{\lambda^h}{1-e^{-\lambda^h \eta k}} \geq \frac{g-r}{\beta}$ and $\frac{\lambda^l}{1-e^{-\lambda^l \eta k}} \geq \frac{g-r}{\beta}$. The third case to be checked is when both the optimal τ^{h*} and τ^{l*} are strictly positive. In this case, solving the system of equations would lead to the optimal exit time for the two types of agents: $\tau^{h*} = \frac{\eta k}{1-\rho} + \tau^{l*} - \frac{1}{(1-\rho)\lambda^h} ln\left(\frac{g-r}{g-r-\beta\lambda^h}\right)$ and $\tau^{l*} = \frac{\eta k}{\rho} + \tau^{h*} - \frac{1}{\rho\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)$. This leads to: $\eta k = (1-\rho)\frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right) + \frac{\rho}{\lambda^h} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)$. Again this is not possible by the assumption made on ρ. Finally it is not possible to have an equilibrium where optimistic agents exit immediately, while pessimistic agents stay for a while invested as in that case the FOC for a pessimist would become: $\frac{\lambda^h}{1-e^{-\lambda^h[\eta k-(1-\rho)\tau^h]}} = \frac{g-r}{\beta}$. Consequently, by assumption i), for which $\frac{\lambda^h}{1-e^{-\lambda^h}} > \frac{g-r}{\beta}$, a pessimist would prefer to exit immediately. Thus, there exists a unique equilibrium, symmetric in t_i, where the bubble bursts for exogenous reasons. ■

Proof (Uniqueness of the symmetric equilibrium in Proposition 3).

First, there cannot be an equilibrium where the bubble always bursts for exogenous reasons. Suppose agents believe that the bubble bursts for exogenous reasons under both compositions. Then, the FOC for type (h, t_i) is $\frac{\lambda^h}{1-\frac{1}{e^{\lambda^h(\bar{\tau}-\tau^h)}}} = \frac{g-r}{\beta}$ and for type (l, t_i) is $\frac{\lambda^l}{1-\frac{1}{e^{\lambda^l(\bar{\tau}-\tau^l)}}} = \frac{g-r}{\beta}$. Thus the optimal strategies are $\tau^{h*} = \bar{\tau} - \frac{1}{\lambda^h}\ln\left(\frac{g-r}{g-r-\lambda^h\beta}\right)$ for type (h, t_i), and $\tau^{l*} = \bar{\tau} - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)$ for type (l, t_i). In order for these strategies to lead to an exogenous burst in both worlds it must be true that $t_0 + \eta\kappa + \bar{\tau} - \frac{1}{\lambda^h}\ln\left(\frac{g-r}{g-r-\lambda^h\beta}\right) \geq t_0 + \bar{\tau}$ and $t_0 + \eta\kappa + \rho\left[\bar{\tau} - \frac{1}{\lambda^h}\ln\left(\frac{g-r}{g-r-\lambda^h\beta}\right)\right] + (1-\rho)\left[\bar{\tau} - \frac{1}{\lambda^l}\ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)\right] \geq t_0 + \bar{\tau}$. The first equation would not be satisfied because, by assumption, $\frac{\lambda^h}{1-e^{-\lambda^h\eta k}} > \frac{g-r}{\beta}$. Thus, this type of equilibrium does not exist. Moreover, no equilibrium, with burst for endogenous reasons if the world is pessimistic and burst for exogenous reasons otherwise, can happen, where h-types exit immediately, as in that case l-types exit strategies alone cannot make the bubble burst for exogenous reasons. Third, no other symmetric equilibrium in which the bubble always bursts for endogenous reasons is possible. Suppose there exists such an equilibrium. There could be two cases: in the first one, the h-types exit as soon as they become aware of the mispricing. In the second case they stay invested for a while. Let us start from the case, where $\tau^h* = 0$. Then the FOC for an h-type must satisfy: $\frac{\lambda^h}{1-\frac{1}{\alpha_1 e^{\lambda^h \eta k}+(1-\alpha_1)e^{\lambda^h[\eta k+(1-\rho)\tau^l)]}}} \geq \frac{g-r}{\beta}$. The FOC for an l-type becomes: $\frac{\lambda^l}{1-e^{-\lambda^l(\eta k-\rho\tau^l)}} = \frac{g-r}{\beta}$. By assumption, the LHS starts below the RHS and it is increasing in τ^l. Thus, the optimal strategy for an l-type is to exit $\tau^{l*} = \frac{1}{\rho}[\eta k - \frac{1}{\lambda^l}ln(\frac{g-r}{g-r-\beta\lambda^l})]$ periods after he becomes aware of the mispricing. Substituting this optimal strategy into the FOC for an h-type leads to: $\frac{\lambda^h}{1-\frac{1}{\alpha_1 e^{\lambda^h \eta k}+(1-\alpha_1)e^{\lambda^h \frac{1}{\rho}\left[\eta k-(1-\rho)\frac{1}{\lambda^l}ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right]}}} \geq \frac{g-r}{\beta}$. By the assumption on $\alpha_1 < \bar{\alpha}$, this is not possible. Thus, if another equilibrium with burst for endogenous reasons in every state of the world exists, it must be such that both types of agents stay invested for a while before exiting the market. Let us consider such an equilibrium. If both types use a strategy that require them to stay invested for a while, then the first order conditions become: $\frac{\lambda^h}{1-\frac{1}{\alpha_1 e^{\lambda^h \eta k}+(1-\alpha_1)e^{\lambda^h[\eta k-(1-\rho)\tau^h+(1-\rho)\tau^l)]}}} = \frac{g-r}{\beta}$ and $\frac{\lambda^l}{1-e^{-\lambda^l(\eta k+\rho\tau^h-\rho\tau^l)}} = \frac{g-r}{\beta}$. The system is satisfied if and only if $\alpha_1 = \bar{\alpha}$. By assumption, this is never the case. Thus, the unique symmetric equilibrium is the one where the bubble bursts for endogenous reasons if the world is pessimistic, and for exogenous reasons otherwise. ■

Proof (Uniqueness of the symmetric equilibrium in Proposition 4).

First, for the same reason of when $\alpha_1 < \bar{\alpha}$, there cannot be another equilibrium where the bubble bursts for exogenous reasons in both world. Second, if the bubble is going to burst for endogenous reasons in both case, h-types always exit as soon as they become aware of the bubble. Suppose that there exists such an equilibrium where h-types stay for a while. Then, these two FOCs must be satisfied: $\frac{\lambda^h}{1-\frac{1}{\alpha_1 e^{\lambda^h \eta k}+(1-\alpha_1)e^{\lambda^h[\eta k-(1-\rho)\tau^h+(1-\rho)\tau^l)]}}} = \frac{g-r}{\beta}$, and $\frac{\lambda^l}{1-e^{-\lambda^l(\eta k+\rho\tau^h-\rho\tau^l)}} = \frac{g-r}{\beta}$. The system is satisfied if and only if $\alpha_1 = \bar{\alpha}$. Thus, if in both worlds the burst happens for endogenous reasons, pessimistic agents exit immediately.

Moreover, there cannot be another equilibrium where the bubble bursts for endogenous reason if the world is pessimistic, and for exogenous reasons otherwise. If such an equilibrium exists, it must be $\tau^h > 0$. If that is the case, then for every type (h, t_i) the optimal strategy would be $\tau^h = \bar{\tau} - \frac{1}{\lambda^h}ln\left[\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)} - \frac{\alpha_1}{1-\alpha_1}e^{\lambda^h\eta k}\right]$, and for every type (l,t_i) $\tau^l = \bar{\tau} - \frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)$. When $\alpha_1 > \bar{\alpha}$ these strategies cannot lead to a burst for exogenous reasons in case the world is populated by a fraction of optimists. Thus, when $\alpha_1 > \bar{\alpha}$, in the unique equilibrium, pessimistic agents exit as soon as they become aware of the bubble, while optimists stay for a while, and the bubble bursts for endogenous reasons in both worlds. ■

Proposition 6 *When $\alpha_1 = \bar{\alpha}$, there exist infinitely many t_i-symmetric perfect Bayesian equilibria where the bubble bursts for endogenous reasons both if the world is pessimistic and if the world is populated also by optimists. In this case, pessimistic agents stay invested: $\tau^h \in \left[0, \bar{\tau} - \frac{1}{\rho}\left[\eta k - \frac{1-\rho}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right]\right)$ periods after they become aware of the mispricing. Optimists stay $\tau^l = \tau^h + \frac{1}{\rho}\left[\eta k - \frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right]$ periods after they become aware of the mispricing. Moreover, there exists also the t_i-symmetric perfect Bayesian equilibrium described in Proposition 3, where the bubble bursts for endogenous reasons if the world is pessimistic, and for exogenous reasons otherwise.*

Proof. In order to have an equilibrium where the bubble always bursts for endogenous reasons we must solve the two equations: $\frac{\lambda^h}{1-\frac{1}{\alpha_1 e^{\lambda^h \eta k}+(1-\alpha_1)e^{\lambda^h[\eta k-(1-\rho)\tau^h+(1-\rho)\tau^l)]}}} = \frac{g-r}{\beta}$, and $\frac{\lambda^l}{1-e^{-\lambda^l(\eta k+\rho\tau^h-\rho\tau^l)}} = \frac{g-r}{\beta}$. This leads to: $\tau^h = \tau^l - \frac{1}{1-\rho}\left[\frac{1}{\lambda^h} ln\left(\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)} - \frac{\alpha_1}{1-\alpha_1}e^{\lambda^h\eta k}\right) - \eta k\right]$, and $\tau^l = \tau^h + \frac{1}{\rho}\left[\eta k - \frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right]$. This system of equations has infinitely many solutions whenever $\alpha_1 = \bar{\alpha}$. In order for agents to have correct beliefs, only the subset of strategies that lead to a burst for endogenous reasons in both worlds can be considered. If the world is pessimistic, the bubble always bursts for endogenous reasons. Thus, the optimal pair of strategies must satisfy only the following constraint: $t_0 + \eta k + \rho\tau^h + (1-\rho)\tau^l < t_0 + \bar{\tau}$. Substituting the optimal τ^l, in response to τ^h, leads to: $\tau^h < \bar{\tau} - \frac{1}{\rho}\left[\eta k - \frac{1-\rho}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right]$. Thus, every strategy of h-types lower than this value, associated with a strategy for an l-type equal to $\tau^h + \frac{1}{\rho}\left[\eta k - \frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right]$, leads to a symmetric equilibrium where the bubble bursts for endogenous reasons in both worlds. For the other equilibrium, if pessimistic agents believe the bubble bursts for endogenous reasons when the world is pessimistic, and for exogenous reasons otherwise, then they think that $\xi = \eta k + \tau^h$ and $\xi' = \bar{\tau}$. Substituting these terms in the FOC leads to $\frac{\lambda^h}{1-\frac{1}{\alpha_1 e^{\lambda^h \eta k}+(1-\alpha_1)e^{\lambda^h(\bar{\tau}-\tau^h)}}} = \frac{g-r}{\beta}$. Since this is valid for every t_i, every h-type waits the same number of periods τ^h after having become aware of the mispricing to sell the asset. Since $\alpha_1 = \bar{\alpha}$, the LHS of the above equation starts below the RHS and is strictly increasing in τ^h. Thus, $\tau^h = \bar{\tau} - \frac{1}{\lambda^h} ln \left[\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)} - \frac{\alpha_1}{1-\alpha_1} e^{\lambda^h \eta k}\right]$. For type (l, t_i) from the FOC and since $\xi' = \bar{\tau}$: $\frac{\lambda^l}{1-e^{-\lambda^l(\bar{\tau}-\tau^l)}} = \frac{g-r}{\beta}$. Thus, every l-type waits the same number of periods τ^l after having become aware of the mispricing before selling the asset: $\tau^l = \bar{\tau} - \frac{1}{\lambda^l} ln \left(\frac{g-r}{g-r-\lambda^l\beta}\right)$. These strategies must lead to an equilibrium where the bubble bursts for endogenous reasons if the world is populated only by pessimists, and for exogenous reasons otherwise: $t_0 + \eta\kappa + \tau^h < t_0 + \bar{\tau}$ and $t_0 + \eta\kappa + \rho\tau^h + (1-\rho)\tau^l \geq t_0 + \bar{\tau}$. Substituting the optimal τ^h and τ^l in the above conditions leads to: $e^{\lambda^h \eta k} < \frac{g-r}{g-r-\beta\lambda^h}$ and $\eta k - \frac{(1-\rho)}{\lambda^l} ln \left(\frac{g-r}{g-r-\beta\lambda^l}\right) \geq \frac{\rho}{\lambda^h} ln \left[\frac{g-r}{(1-\alpha_1)(g-r-\beta\lambda^h)} - \frac{\alpha_1}{1-\alpha_1} e^{\lambda^h \eta k}\right]$. The first inequality is true by assumption. If $\alpha_1 = \overline{\alpha}$, also the second inequality is satisfied. Finally, no other symmetric equilibria where the bubble bursts for exogenous reasons in both worlds can exist when $\alpha_1 = \overline{\alpha}$, as in that case the strategies of pessimistic agents cannot lead to a burst for exogenous reasons when the world is pessimistic. Similarly, it does not exist a symmetric equilibrium where $\tau^h = 0$ and the bubble bursts for endogenous reasons if the world is pessimistic, and for exogenous reasons otherwise. Indeed, in that case the strategies of the optimistic agents alone cannot lead to a burst for exogenous reasons. ■

Lemma 1 *When $\alpha_1 = \bar{\alpha}$, the equilibrium where the bubble bursts for endogenous reasons if the world is pessimistic, and for exogenous reasons otherwise, is ex ante Pareto dominant.*

Proof. First, I show that all the equilibria with burst for endogenous reasons in both worlds can be ranked in terms of the expected utility they give to agents, from the lowest, where $\tau^h = 0$, to the highest, where $\tau^h = \bar{\tau} - \frac{1}{\rho}\left[\eta k - \frac{1-\rho}{\lambda^l} ln \left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right] - \epsilon$. As seen above, for a pessimistic agent t_i, the expected utility from exiting at time t is: $\int_{t_i}^{t} e^{-rs}(1-\beta)p(s)\pi^h(s \mid t_i)ds + e^{-rt}p(t)(1-\Pi^h(t \mid t_i))$.
In case the equilibrium features a bubble that bursts for endogenous reasons in both

worlds, $\xi = \eta k + \tau^h$ and $\xi' = \eta k + \rho\tau^h + (1-\rho)\tau^l$. Substituting the equilibrium optimal strategies τ^{h*} and $\tau^{l*} = \tau^{h*} + \frac{1}{\rho}\left[\eta k - \frac{1}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right]$, leads to $\xi = \eta k + \tau^{h*}$ and $\xi' = \frac{\eta k}{\rho} + \tau^{h*} - \frac{(1-\rho)}{\rho\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)$. Thus, the equilibrium expected utility for type t_i, in function of τ^{h*}, is:

$$\int_{t_i}^{t_i+\tau^{h*}} \frac{(1-\beta)p(t_i+s)\lambda^h}{e^{r(t_i+s)}(e^{\lambda^h\eta}-1)}\left[\alpha_1 e^{\lambda^h(\eta k+\tau^{h*}-s)} + (1-\alpha_1)e^{\lambda^h\left(\frac{\eta k}{\rho}+\tau^{h*}-\frac{(1-\rho)}{\rho\lambda^l}ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)-s\right)}\right]ds+$$
$$+\frac{e^{-r(t_i+\tau^{h*})}p(t_i+\tau^{h*})\left[\alpha_1(e^{\lambda^h\eta k}-1)+(1-\alpha_1)\left(e^{\lambda^h\left(\frac{\eta k}{\rho}-\frac{(1-\rho)}{\rho\lambda^l}ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right)}-1\right)\right]}{e^{\lambda^h\eta}-1}.$$

This equilibrium expected utility is increasing in τ^{h*}, as its terms increase in it. Similarly, for an optimistic t_i, the expected utility in function of τ^{l*} in an equilibrium with burst for endogenous reasons in both world is:

$$\int_{t_i}^{t_i+\tau^{l*}} \frac{(1-\beta)p(t_i+s)\lambda^l}{e^{r(t_i+s)}(e^{\lambda^l\eta}-1)} e^{\lambda^l\left[\frac{\eta k}{1-\rho}+\tau^{l*}-\frac{\rho}{(1-\rho)\lambda^h}ln\left(\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)}-\frac{\alpha_1}{1-\alpha_1}e^{\lambda^h\eta k}\right)-s\right]}ds +$$
$$+\frac{e^{-r(t_i+\tau^{l*})}p(t_i+\tau^{l*})\left[e^{\lambda^l\left(\frac{\eta k}{1-\rho}-\frac{\rho}{(1-\rho)\lambda^h}ln\left(\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)}-\frac{\alpha_1}{1-\alpha_1}e^{\lambda^h\eta k}\right)\right)}-1\right]}{e^{\lambda^l\eta}-1}.$$

The equilibrium expected utility for an optimist is increasing in τ^{l*}, as its terms increase in it. Then, the symmetric equilibria with bursts for endogenous reasons can be ranked according to the expected utility they give to agents, from the lowest, where a pessimistic agent exits as soon as he knows there is a mispricing, to the highest where he stays $\tau^h = \bar{\tau} - \frac{1}{\rho}\left[\eta k - \frac{1-\rho}{\lambda^l} ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right] - \epsilon$. The next step is to show that the highest possible symmetric equilibrium with endogenous bursts in both worlds is Pareto dominated by the symmetric equilibrium with burst for endogenous reasons if the world is pessimistic, and for exogenous reasons otherwise. Thus, I need to compare the two equilibrium expected utilities both for h-types and l-types. For an h-type, the expected utility in the equilibrium where the bubble always bursts for endogenous reasons is:

$$\int_{t_i}^{t_i+\tau^{h*}} \frac{(1-\beta)p(t_i+s)\lambda^h}{e^{r(t_i+s)}(e^{\lambda^h\eta}-1)}\left[\alpha_1 e^{\lambda^h(\eta k+\tau^{h*}-s)} + (1-\alpha_1)e^{\lambda^h\left(\frac{\eta k}{\rho}+\tau^{h*}-\frac{(1-\rho)}{\rho\lambda^l}ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)-s\right)}\right]ds+$$

$$+\frac{e^{-r(t_i+\tau^{h*})}p(t_i+\tau^{h*})\left[\alpha_1(e^{\lambda^h\eta k}-1)+(1-\alpha_1)\left(e^{\lambda^h(\frac{\eta k}{\rho}-\frac{(1-\rho)}{\rho\lambda^l}ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)}-1\right)\right]}{e^{\lambda^h\eta}-1},$$

where $\tau^{h*}=\bar{\tau}-\frac{1}{\rho}\left[\eta k-\frac{1-\rho}{\lambda^l}ln\left(\frac{g-r}{g-r-\beta\lambda^l}\right)\right]-\epsilon$. Instead, his expected utility in the equilibrium where the bubble bursts for exogenous reasons when the world is populated also by optimists is:

$$\int_{t_i}^{t_i+\bar{\tau}^{h*}}\frac{e^{-r(t_i+s)}(1-\beta)p(t_i+s)\lambda^h}{e^{\lambda^h\eta}-1}\left[\alpha_1 e^{\lambda^h(\eta k+\bar{\tau}^{h*}-s)}+(1-\alpha_1)e^{\lambda^h(\bar{\tau}-s)}\right]ds+$$
$$+\frac{e^{-r(t_i+\bar{\tau}^{h*})}p(t_i+\bar{\tau}^{h*})\left[\alpha_1(e^{\lambda^h\eta k}-1)+(1-\alpha_1)\left(e^{ln\left(\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)}-\frac{\alpha_1}{1-\alpha_1}e^{\lambda^h\eta k}\right)}-1\right)\right]}{e^{\lambda^h\eta}-1},$$

where $\bar{\tau}^{h*}=\bar{\tau}-\frac{1}{\lambda^h}ln\left[\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)}-\frac{\alpha_1}{1-\alpha_1}e^{\lambda^h\eta k}\right]$. Since $\bar{\tau}^{h*}>\tau^{h*}$, and $\alpha_1=\bar{\alpha}$, the second expected utility is always greater than the first one. Similarly, for an optimistic t_i, the expected utility in function of τ^{l*} in an equilibrium with burst for endogenous reasons is:

$$\int_{t_i}^{t_i+\tau^{l*}}\frac{e^{-r(t_i+s)}(1-\beta)p(t_i+s)\lambda^l}{e^{\lambda^l\eta}-1}e^{\lambda^l\left(\frac{\eta k}{1-\rho}+\tau^{l*}-\frac{\rho}{(1-\rho)\lambda^h}ln\left(\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)}-\frac{\alpha_1}{1-\alpha_1}e^{\lambda^h\eta k}\right)-s\right)}ds+$$
$$+\frac{e^{-r(t_i+\tau^{l*})}p(t_i+\tau^{l*})\left[e^{\lambda^l(\frac{\eta k}{1-\rho}-\frac{\rho}{(1-\rho)\lambda^h}ln\left(\frac{g-r}{(1-\alpha_1)(g-r-\lambda^h\beta)}-\frac{\alpha_1}{1-\alpha_1}e^{\lambda^h\eta k}\right)}-1\right]}{e^{\lambda^l\eta}-1},$$

where $\tau^{l*}=\bar{\tau}-\frac{1}{\lambda^l}ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)-\epsilon$. Instead, his expected utility in the equilibrium where the bubble bursts for exogenous reasons is:

$$\int_{t_i}^{t_i+\bar{\tau}^{l*}}\frac{e^{-r(t_i+s)}(1-\beta)p(t_i+s)\lambda^l}{e^{\lambda^l\eta}-1}e^{\lambda^l(\bar{\tau}-s)}ds+\frac{e^{-r(t_i+\bar{\tau}^{l*})}p(t_i+\bar{\tau}^{l*})\left[\frac{g-r}{g-r-\beta\lambda^l}-1\right]}{e^{\lambda^l\eta}-1},$$

where $\bar{\tau}^{l*}=\bar{\tau}-\frac{1}{\lambda^l}ln\left(\frac{g-r}{g-r-\lambda^l\beta}\right)$. Since $\bar{\tau}^{l*}>\tau^{l*}$, and $\alpha_1=\bar{\alpha}$, the second expected utility is greater than the first. Thus, when $\alpha_1=\bar{\alpha}$, the equilibrium where the bubble bursts for endogenous reasons in a pessimistic world, and for exogenous reasons otherwise, Pareto dominates all the possible symmetric equilibria. ■